GRAPH THEORY

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These notes have been prepared for students of Math 352 at Philadelphia University, Jordan.[1] Outline notes are not meant for self-study; No student is expected to fully benefit from these notes unless they have regularly attended the lectures.

1 Definitions

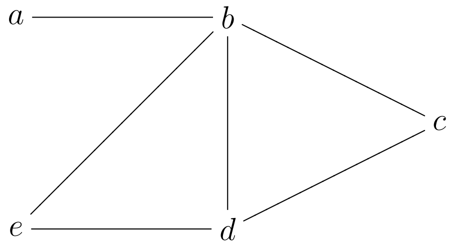
Definition. A *graph* G is a composite of two finite sets which are commonly labeled $V_G = \{v_1, v_2, \ldots, v_n\}$ and $E_G = \{e_1, e_2, \ldots, e_m\}$. Elements of V_G are called *vertices*, and elements of E_G *edges*. An edge e is actually a set of exactly two vertices, e.g., we may write $e = \{v_1, v_2\}$ or simply $e = v_1v_2$.

When $ab \in E_G$, we say that the vertex a is *adjacent* to b (hence also b to a). For every vertex $a \in V_G$, we define the *neighbors* of a to be the set of all vertices which are adjacent to a, i.e., $N(a) = \{b \in V_G \mid ab \in E_G\}$. The number of neighbors is the *degree* of the vertex, i.e., $\deg(a) = |N(a)|$. Then, we define

$$\Delta(G) = \max_{a \in V_G} \deg(a) \quad \text{and} \quad \deg G = \sum_{a \in V_G} \deg(a)$$

Also, the neighbors of a set of vertices $S \subseteq V_G$ is given by $N(S) = \bigcup_{a \in S} N(a)$

Example. Let $V_G = \{a, b, c, d, e\}$ and $E_G = \{ab, bc, bd, be, cd, de\}$. This G can be represented by a picture (i.e., a graph) e.g.,



Observe that $N(a) = \{b\}$ and $N(b) = \{a, c, d, e\}$, and $N(\{d, e\}) = \{b, c, d, e\}$. In this case we have $\deg(a) = 1$, $\deg(b) = 4$, $\deg(c) = 2$, $\deg(d) = 3$, $\deg(e) = 2$, hence $\deg G = 1 + 4 + 2 + 3 + 2 = 12$. Note here that $\Delta(G) = 4$.

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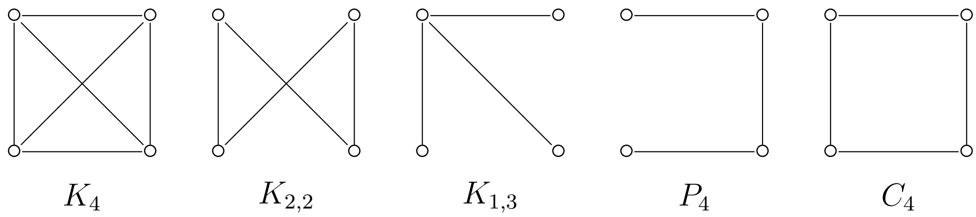
Theorem 1 (Euler's Theorem)**.** The degree of any graph G is twice the number of its edges, i.e., $\deg G = 2\,|E_G|$. In particular, the degree of any graph is an even number.

Proof. Every edge $ab \in E_G$ contributes two to the degree of G, one via $\deg(a)$ and one via $\deg(b)$. Hence the summation of degrees equals twice the summation of edges. $\triangledown$

Definition. We call G a *trivial graph* when $E_G = \emptyset$, i.e., when G has only vertices with no edges. Other families of special graphs are now introduced:

1. A *complete graph* K_n is a graph of n vertices, all of which are adjacent one to another. In particular, K_3 is also called a *triangle.*
2. A *complete bipartite graph* $K_{m,n}$ consists of $m+n$ vertices that are partitioned into two subsets, with m and n elements in each, such that two vertices are adjacent if and only if they do not belong in the same subset.
3. A *path* P_n $(n \geq 2)$ is a graph with $V_{P_n} = \{v_1, v_2, \ldots, v_n\}$ and $E_{P_n} = \{e_1, e_2, \ldots, e_{n-1}\}$, where each $e_i = v_i v_{i+1}$. In this notation, we say that P_n is a path from v_1 to v_n.
4. A *cycle* C_n $(n \geq 3)$ is obtained from the path P_n by adding one more edge: $v_n v_1$. Hence, C_n is also called a *closed path* of n vertices.

Example.



Definition. A graph is called *regular* if all its vertices have equal degrees. In particular, when a graph G has $\deg(a) = d$ for all $a \in V_G$, then we say G is *d-regular.* If a graph is not regular, then it is *irregular.*

Example. The graph K_4 is regular, all cycles are 2-regular, and $K_{2,3}$ is irregular.

Definition. If $V_G = \{v_1, v_2, \ldots, v_n\}$, the *degree sequence* of the graph G is the sequence $(\deg(v_i))$ of length n, arranged in (weakly) decreasing order. We call a random decreasing sequence of positive integers *graphical* if we can find a graph with this degree sequence.

Example. The degree sequence of P_5 is $(2,2,2,1,1)$, hence the sequence $(2,2,2,1,1)$ is graphical. On the other hand, the sequence $(4,3,2,2,2)$ is not graphical, because it is impossible to have a graph G with odd $\deg G = 4+3+2+2+2 = 13$. (Why?)

Algorithm 2 (Graphical Degree Sequence)**.** Given a decreasing sequence $(d_1, d_2, \ldots, d_n)$ of positive integers, we determine graphical or not graphical.

1. Delete the first integer, say k. (Initially $k = d_1$.)
2. From what remains, subtract the first k numbers each by one. If we get a negative number, the sequence is not graphical. If we get all zeros, the sequence is graphical.
3. Rearrange the resulting sequence in decreasing order, if necessary, and then repeat the above two steps until a conclusion is obtained.

Example. We illustrate the algorithm on the sequence $(3,2,2,1,1,1)$:

$$(3,2,2,1,1,1) \to (1,1,0,1,1) \to (1,1,1,1,0) \to (0,1,1,0) \to (1,1,0,0) \to (0,0,0)$$

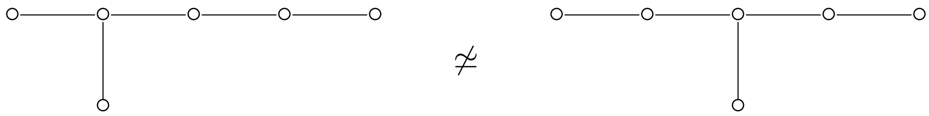
We finish with all zeros, so the sequence is graphical.

Definition. Two graphs G and H are *isomorphic* to each other, written $G \simeq H$, if there exists a bijection $f : V_G \to V_H$ such that $ab \in E_G$ if and only if $f(a)f(b) \in E_H$ for all $a, b \in V_G$.

Example. Note that $K_3 \simeq C_3$ (both are triangles) and that $K_2 \simeq K_{1,1} \simeq P_2$.

In essence, $G \simeq H$ if and only if both graphs can be represented by identical pictures. Hence it is necessary, but not sufficient, for G and H to have the same number of vertices, the same number of edges, the same degree sequences, etc.

Example. Two non-isomorphic graphs can have identical degree sequence $(3,2,2,1,1,1)$.



To prove that they are not isomorphic to each other, note that each graph has a unique vertex of degree 3, call them a and w. So any bijection f must have $f(a) = w$ to be an isomorphism. This is impossible since $N(a)$ consists of vertices with degree sequence $(2,1,1)$, whereas for $N(w)$ we have $(2,2,1)$.

Definition. A graph H is a *subgraph* of the graph G if $V_H \subseteq V_G$ and $E_H \subseteq E_G$. We write $H \subseteq G$ and may say that G *contains* H. By abuse of notation, we also write $H \subseteq G$ when we mean that G contains a subgraph which is isomorphic to H.

Example. Note that $P_3 \subseteq C_3$, $K_{2,2} \subseteq K_{2,4}$, $C_3 \not\subseteq K_{2,4}$, and $K_4 \subseteq K_5$.

Definition. A graph G is *connected* if there is a path from any vertex to any other vertex in G, otherwise *disconnected*. A *component* of G is a maximal connected subgraph of G. Hence, a graph is disconnected if and only it has more than one component.

Definition. An edge of G is called a *bridge* if removing it would increase the number of components of G. So if G is connected, removing a bridge would make G disconnected.

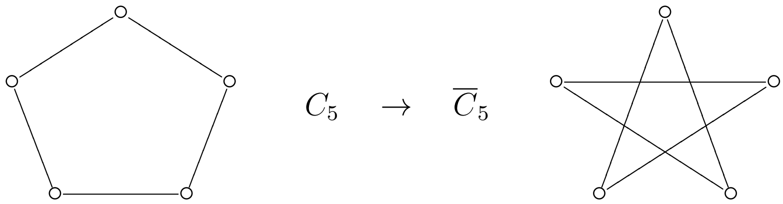
Example. Note that in P_4 every edge is a bridge, whereas C_4 has no bridge.

Theorem 3. Let G be a connected graph with n vertices. Then there exists a connected subgraph H such that $|V_H| = n$ and $|E_H| = n - 1$. Hence, if any graph G is connected, then $|E_G| \geq |V_G| - 1$.

Proof. Assume $n \geq 2$, so we can have a subgraph with 2 vertices and the edge v_1v_2. Since G is connected, if $n \geq 3$, then there is another vertex $v_3 \in N(\{v_1, v_2\})$, i.e., with either v_1v_3 or $v_2v_3 \in E_G$, so we can have a connected subgraph with 3 vertices and 2 edges. And if $n \geq 4$, there exists $v_4 \in N(\{v_1, v_2, v_3\})$ with v_1v_4, v_2v_4, or $v_3v_4 \in E_G$, so now we can have a connected subgraph with 4 vertices and 3 edges. Repeating this process will produce a connected subgraph H with $V_H = V_G$ and $|E_H| = n - 1$. $\triangledown$

Definition. The *complement* of a graph G is the graph $\overline{G}$, where $V_{\overline{G}} = V_G$ and $ab \in E_{\overline{G}}$ if and only if $ab \notin E_G$.

Example. We show here the picture of C_5 next to its complement.



Theorem 4. If G is disconnected, then $\overline{G}$ is connected.

Proof. Suppose that G is disconnected and consider two vertices a and b. Remember that $V_{\overline{G}} = V_G$. We will show there is a path from a to b in $\overline{G}$.

If a and b are not adjacent in G, then they are in $\overline{G}$, so such path (the edge ab) exists. If $ab \in E_G$, choose a vertex c belonging to a component of G not containing a, b. Such a vertex c exists since G is disconnected. Then $ac \notin E_G$ and $cb \notin E_G$, and we have the path ac, cb in $\overline{G}$. ▽

Definition. A graph G is *self-complementary* when $\overline{G} \simeq G$. The preceding theorem says that a self-complementary graph must be connected.

Example. Observe from the previous example that $C_5 \simeq \overline{C}_5$. Another example is P_4.

Recall from Linear Algebra that if M denotes an arbitrary matrix, we write $[M]_{ij}$ to refer to the (i, j) entry in M, i.e., the entry in the ith row and jth column of M.

Definition. Suppose that $V_G = \{v_1, v_2, \ldots, v_n\}$. The *adjacency matrix* of the graph G is the $n \times n$ matrix A given by

$$[A]_{ij} = \begin{cases} 1 & \text{if} \quad v_i v_j \in E_G \\ 0 & \text{if} \quad v_i v_j \notin E_G \end{cases}$$

Example. The adjacency matrix of C_4, where $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$, is given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

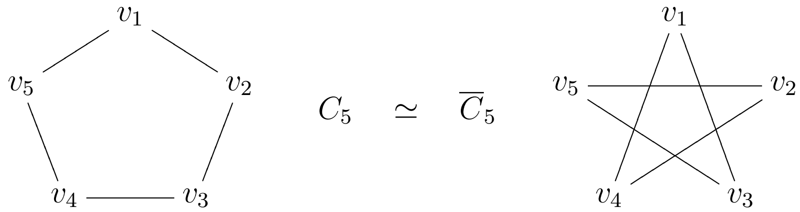
Definition. A *permutation matrix* is a square matrix obtained from the identity matrix by reordering its rows. A known fact is that every permutation matrix P belongs to the family of orthogonal matrices, i.e., that $P^{-1} = P^T$.

Example. Let P be the permutation matrix obtained from the identity matrix by permuting its rows in the order of $(1, 3, 5, 2, 4)$, i.e.,

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Theorem 5. Let the graphs G and H have adjacency matrices A and B, respectively. Then $G \simeq H$ if and only if $A = PBP^T$ for some permutation matrix P.

Example. We look again at the fact that $C_5 \simeq \overline{C}_5$, this time with labeled vertices.



An obvious bijection is one which permutes the vertex indices $(1, 2, 3, 4, 5)$ to $(1, 3, 5, 2, 4)$, thus the permutation matrix P given earlier. Denote the corresponding adjacency matrices A and B, respectively, and verify that $A = PBP^T$:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Definition. Suppose that $V_G = \{v_1, v_2, \ldots, v_n\}$ and $E_G = \{e_1, e_2, \ldots, e_m\}$. Then the *incidence matrix* of the graph G is the $n \times m$ matrix Z given by

$$[Z]_{ij} = \begin{cases} 1 & \text{if} \quad v_i \in e_j \\ 0 & \text{if} \quad v_i \notin e_j \end{cases}$$

Example. The incidence matrix of P_4, where $E = \{v_1v_2, v_2v_3, v_3v_4\}$, is given by

$$Z = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition. Let G be a graph with vertices $v_1, v_2, \ldots, v_n$. The *degree matrix* of G is the $n \times n$ diagonal matrix D given by $[D]_{ii} = \deg(v_i)$.

Theorem 6. Suppose that the adjacency matrix A and incidence matrix Z have been given for the same graph G. Then $ZZ^T = A + D$, where D is the degree matrix of G.

Proof. For $i \neq j$, we have

$$[ZZ^T]_{ij} = \sum_{k \geq 1} [Z]_{ik}[Z^T]_{kj} = \sum_{k \geq 1} [Z]_{ik}[Z]_{jk}$$

If $v_iv_j \in E$, then there is exactly one value of k for which $[Z]_{ik} = [Z]_{jk} = 1$, while the rest either $[Z]_{ik} = 0$ or $[Z]_{jk} = 0$. In that case, both ZZ^T and $A + D$ have 1 in their (i, j) entries. If $v_iv_j \notin E$, no such k exists, and that entry will be 0 in both.

For $i = j$, we have

$$[ZZ^T]_{ii} = \sum_{k \geq 1} [Z]_{ik}[Z^T]_{ki} = \sum_{k \geq 1} [Z]_{ik}^2 = \sum_{k \geq 1} [Z]_{ik}$$

(Note the exponent 2 is redundant when squaring 0 or 1.) So $[ZZ^T]_{ii}$ counts the number of vertices adjacent to v_i, which agrees with the diagonal entry $[A + D]_{ii} = [D]_{ii}$. ▽

2 Trees

Definition. A graph G is *cyclic* if there exists $C_n \subseteq G$ for any $n \geq 3$. A graph which contains no cycle is called *acyclic.* A *tree* is a connected acyclic graph.

Example. K_4 is a cyclic graph. P_4 is acyclic and connected, hence P_4 is a tree.

Theorem 7. Let G be a connected graph. The following are all equivalent.

1. G is acyclic.

2. Every edge in G is a bridge.

3. The size of G is determined by $|E_G| = |V_G| - 1$.

4. There is a unique path between any two vertices in G.

Hence, a connected graph is a tree if and only if any one of the above conditions holds.

Proof. We will show $\neg(4) \to \neg(1) \to \neg(2) \to \neg(3)$: Being connected, we have a path from any vertex a to b. If there are two such paths, their union is either a cycle or contains a cycle, hence cyclic. Moreover, every edge belonging to a cycle is a non-bridge. And if a non-bridge edge is removed, the resulting subgraph H remains connected. By Theorem 3, we have $|E_H| \geq |V_H| - 1 = |V_G| - 1$. Since $|E_G| = |E_H| + 1$, then $|E_G| > |V_G| - 1$.

The last case is $\neg(3) \to \neg(4)$: Assume $|E_G| \neq |V_G| - 1$. By Theorem 3, we have $|E_G| > |V_G| - 1$ and there is a connected subgraph H with $V_H = V_G$ and $|E_H| = |V_G| - 1$. Hence there is an edge $ab \in E_G$ but $ab \notin E_H$. However, H connected means that we already have a path from a to b in H. Thus the edge ab is a second such path in G. $\triangledown$

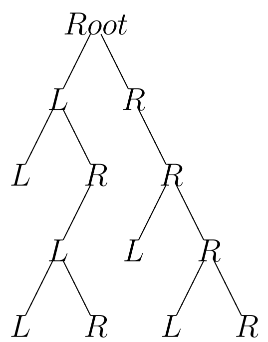
Definition. In a graph, a vertex of degree one is called a *leaf.*

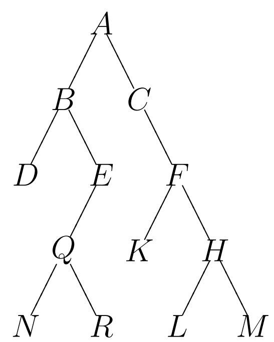
Theorem 8. Every tree has at least two leaves. More precisely, if n_i denotes the number of vertices of degree i, then every tree has $2 + \sum_{i \geq 3}(i-2)n_i$ leaves.

Proof. Let $(d_1, d_2, \ldots, d_n)$ be the degree sequence. For trees, $d_n \geq 1$ since connected and $\sum d_i = 2(n-1)$, so the number of leaves is minimum in the case $(2, 2, \ldots, 2, 1, 1)$. The existence of a vertex of degree $i \geq 3$ will reduce the number of 2s and increase the number of 1s—exactly $i - 2$ of them. $\triangledown$

Definition. Any vertex of a tree can be designated as the *root* with respect to which every other vertex is pictorially represented below the root from which, by uniqueness of path, each level down reflects the length of the path to each vertex. A *binary tree* is a rooted tree in which every vertex can have at most two neighbors below it, which are also called *children.* We say a binary tree is *labeled* when each child is specified Left or Right.

Example. A labeled binary tree with vertex A as the root:





Algorithm 9 (Traversal Algorithms for Labeled Binary Trees)**.** We traverse the vertices of a labeled binary tree according to one of these three algorithms.

1. Pre-Order Algorithm: Root → Left → Right
2. Post-Order Algorithm: Left → Right → Root
3. In-Order Algorithm: Left → Root → Right

Example. We show the order of vertex traversal using each algorithm applied on the labeled binary tree given above.

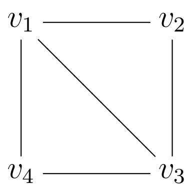
1. Pre-Order: $(A, B, D, E, Q, N, R, C, F, K, H, L, M)$
2. Post-Order: $(D, N, R, Q, E, B, K, L, M, H, F, C, A)$
3. In-Order: $(D, B, N, Q, R, E, A, C, K, F, L, H, M)$

Definition. A *spanning tree* of a graph G is a tree $H \subseteq G$ with $V_H = V_G$. We know, by Theorem 3, that a spanning tree exists if and only if G is connected.

Example. P_4 is a spanning tree of C_4. Another example: any tree with 5 vertices, e.g., P_5 or $K_{1,4}$, is a spanning tree of K_5.

Theorem 10 (The Matrix Tree Theorem)**.** Let G be a connected graph with labeled vertices, adjacency matrix A, and degree matrix D. Then any cofactor of the matrix $D - A$ will give the number of spanning trees of G.

Example. The graph G is given, together with the associated matrix $D - A$.



$$G \quad \to \quad D - A = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

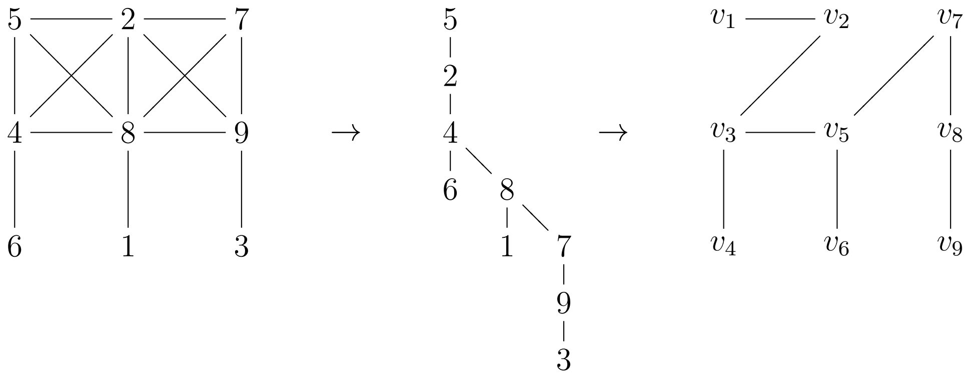
Recall that the cofactor $C_{i,j}$ of a square matrix M is $(-1)^{i+j}$ times the determinant of the matrix obtained from M by removing the ith row and jth column. For example, $C_{3,1} = 8$, i.e., G has 8 spanning trees.

$$C_{3,1} = (+) \det \begin{bmatrix} -1 & -1 & -1 \\ 2 & -1 & 0 \\ 0 & -1 & 2 \end{bmatrix} = 8$$

Algorithm 11 (Depth-First Search)**.** Given a connected graph G with pre-ordered vertices $v_1, v_2, \ldots, v_n$, we produce a rooted spanning tree.

1. Choose a vertex v_k as the root.
2. Directly below this root, place the one adjacent vertex of least index which has not been selected, called this the subroot.
3. Repeat Step 2 with respect to the new subroot. If there is no adjacent vertices left, backtrack upward to the most immediate parent that has an unselected neighbor, placing it in a new column.
4. Repeat until all vertices have been traversed.

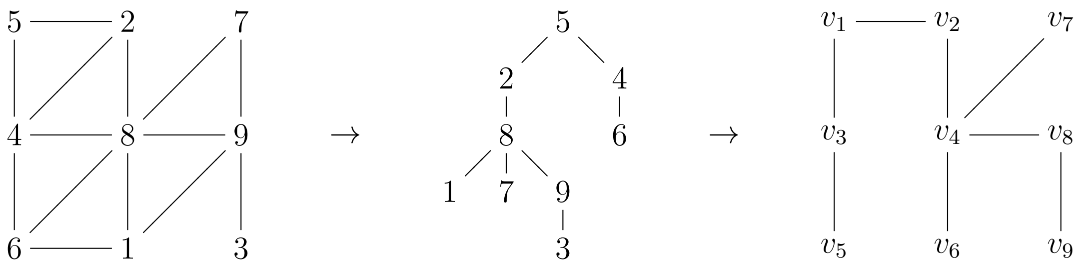
Example. For the given graph, we choose vertex (5) as the root. The DFS spanning tree is shown with the vertex ordering according to the DFS sequence $(5, 2, 4, 6; 8, 1; 7, 9, 3)$.



Algorithm 12 (Breadth-First Search)**.** Given a connected graph G with pre-ordered vertices $v_1, v_2, \ldots, v_n$, we produce a rooted spanning tree.

1. Choose a vertex v_k as the root.
2. Directly below this root, place all the adjacent vertices that have not been selected, from the least index left to right. These are the immediate subroots.
3. Repeat Step 2 with respect to the new subroots, one at a time from left to right.
4. Repeat until all vertices have been traversed.

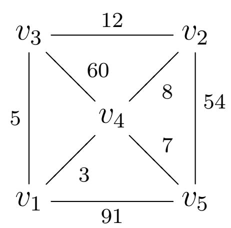
Example. For the given graph, we choose vertex (5) as the root. The BFS spanning tree is shown with the vertex ordering according to the BFS sequence $(5, 2, 4; 8, 6; 1, 7, 9; 3)$.



Definition. A graph is *weighted* when every edge is associated with a positive numerical value, called the *weight*. The *weight matrix* of a weighted graph G is just the adjacency matrix of G in which the weight of each edge is revealed.

Example. A weighted graph G and its weight matrix W:

$G =$



$$\leftrightarrow \quad W = \begin{bmatrix} 0 & 0 & 5 & 3 & 91 \\ 0 & 0 & 12 & 8 & 54 \\ 5 & 12 & 0 & 60 & 0 \\ 3 & 8 & 60 & 0 & 7 \\ 91 & 54 & 0 & 7 & 0 \end{bmatrix}$$

Definition. A *minimal spanning tree* of a weighted graph is a spanning tree with the least total weight. While a minimal spanning tree may not be unique, the least value of its total weight, by definition, is unique.

Algorithm 13 (Kruskal's Algorithm)**.** Given a connected weighted graph G, we generate a minimal spanning tree T.

1. Choose an edge e_1 of least weight to start with $E_T = \{e_1\}$.

2. Choose another edge from E_G of least weight to be added to T, provided that the resulting subgraph remains acyclic.

3. Repeat until $|E_T| = |V_G| - 1$.

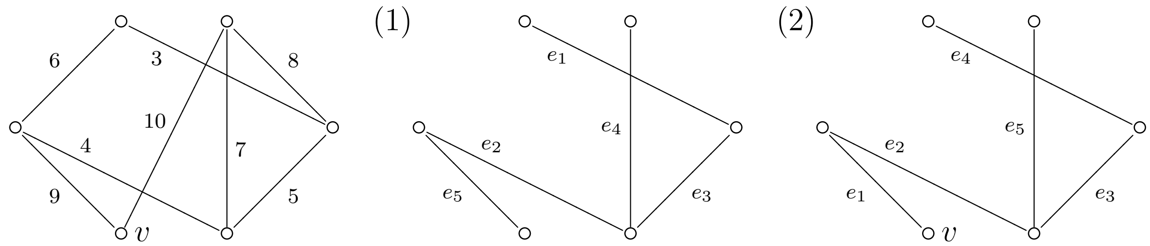
Algorithm 14 (Prim's Algorithm)**.** Given a connected weighted graph G, we generate a minimal spanning tree T, starting at a specified vertex v.

1. Choose an edge containing v with the least weight, say $E_T = \{e_1\}$.

2. Choose another edge from E_G of least weight to be added to T, provided that the resulting subgraph remains acyclic and connected.

3. Repeat until $|E_T| = |V_G| - 1$.

Example. For comparison, we show both Kruskal (1) and Prim (2) for the same given graph. The MST sequence is $(3, 4, 5, 7, 9)$ for Kruskal, and $(9, 4, 5, 3, 7)$ for Prim, both with total weight $= 28$.



3 Walking

Definition. A *walk* of length n is a sequence of edges $v_1v_2, v_2v_3, \ldots, v_nv_{n+1}$. Unless the walk is a path, these edges are not assumed distinct, and neither are the vertices. We have a *closed walk* if $v_{n+1} = v_1$.

Example. In particular, the path P_n is a walk of length $n-1$, and C_n is a closed walk of length n. Another example: $ab, bc, cd, db, be, ec, cd$ is a walk of length 7.

Theorem 15. Let A be the adjacency matrix of a graph G with vertices $v_1, v_2, \ldots, v_n$. The number of walks of length k from v_i to v_j is then given by $[A^k]_{ij}$.

Proof. A walk of length 1 is an edge ij, and we have $[A]_{ij} = 1$ if and only if this edge exists. We proceed by induction. Observe that a walk of length $k+1$ from v_i to v_j consists of a walk of length k from v_i to a neighbor of v_j, call it v_m. The number of such walks is the summation over all such v_m, i.e., $\sum_{m=1}^{n} [A^k]_{im} [A]_{mj} = [A^{k+1}]_{ij}$ as claimed. $\triangledown$

Theorem 16. The number of triangles in G is one-sixth of the trace of A^3, i.e.,

$$\#\triangle = \frac{1}{6} \sum_{i \geq 1} [A^3]_{ii}$$

Proof. A triangle is a closed walk of length 3, so the preceding theorem applies. The division by 6 is due to the 6 different ways to label the vertices of one triangle. ▽

Definition. The *distance* between two vertices a and b, denoted by $d(a,b)$, is the length of the shortest walk from a to b, if it exists; otherwise let $d(a,b)=\infty$.

Example. Note that the shortest walk is necessarily a path, and that $d(a,b)=1$ if and only if a and b are adjacent. It is also clear that $d(a,a)=0$ for any vertex a.

Definition. Suppose that $V_G=\{v_1,v_2,\dots,v_n\}$. The *distance matrix* of the graph G is the $n\times n$ matrix D given by $[D]_{ij}=d(v_i,v_j)$.

Example. The distance matrix of P_4, when labeled in the standard way, is given by

$$D=\begin{bmatrix}0&1&2&3\\1&0&1&2\\2&1&0&1\\3&2&1&0\end{bmatrix}$$

Algorithm 17 (From A to D)**.** With the help of computer, we can retrieve the distance matrix D from the adjacency matrix A.

1. Compute the matrices $A,A^2,A^3,\dots,A^n$, where $n\times n$ is the size of A.
2. Set $[D]_{ii}=0$, and for $i\neq j$ let $[D]_{ij}=k$, the least exponent for which $[A^k]_{ij}\neq 0$. If no such k exists, set $[D]_{ij}=\infty$.

Definition. The *diameter* of a graph G, denoted by $d(G)$, is the largest possible distance between two vertices in G. Thus $d(G)$ is the largest entry in the distance matrix of G.

Example. We have $d(K_4)=1$ and $d(P_4)=3$. In general $d(G)=1$ if and only if G is a complete graph. Note also that $d(G)=\infty$ if and only if G is disconnected.

Theorem 18. If $d(G)\geq 3$, then $d(\overline{G})\leq 3$.

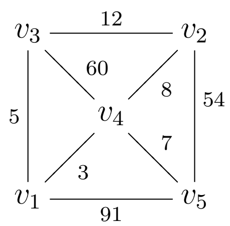
Proof. Assume $d(G)\geq 3$ and let $v,w\in V_G=V_{\overline{G}}$. We will show that $d(v,w)\leq 3$ in $\overline{G}$.

We know there are a and b for which $d(a,b)\geq 3$ in G. It is then false if both $av,vb\in E_G$ or both $aw,wb\in E_G$. So we have four cases to consider:
(1) $E_{\overline{G}}$ contains av and aw, hence $d(v,w)\leq 2$ in $\overline{G}$.
(2) $E_{\overline{G}}$ contains vb and wb, again $d(v,w)\leq 2$ in $\overline{G}$.
(3) $E_{\overline{G}}$ contains av and wb. Since we also have $ab\in E_{\overline{G}}$, then $d(v,w)\leq 3$ in $\overline{G}$.
(4) $E_{\overline{G}}$ contains vb and aw. Again with $ab\in E_{\overline{G}}$, we have $d(v,w)\leq 3$ in $\overline{G}$. ▽

Definition. If G is a weighted graph, we redefine the distance $d(a,b)$ to be the least total weight of all possible walks from a to b.

Example. Given the weight matrix W, we find the distance matrix D.

$$W=\begin{bmatrix}0&0&5&3&91\\0&0&12&8&54\\5&12&0&60&0\\3&8&60&0&7\\91&54&0&7&0\end{bmatrix}\to$$



$$\to D=\begin{bmatrix}0&11&5&3&10\\11&0&12&8&15\\5&12&0&8&15\\3&8&8&0&7\\10&15&15&7&0\end{bmatrix}$$

Note that this weighted graph has diameter $d(G)=15$.

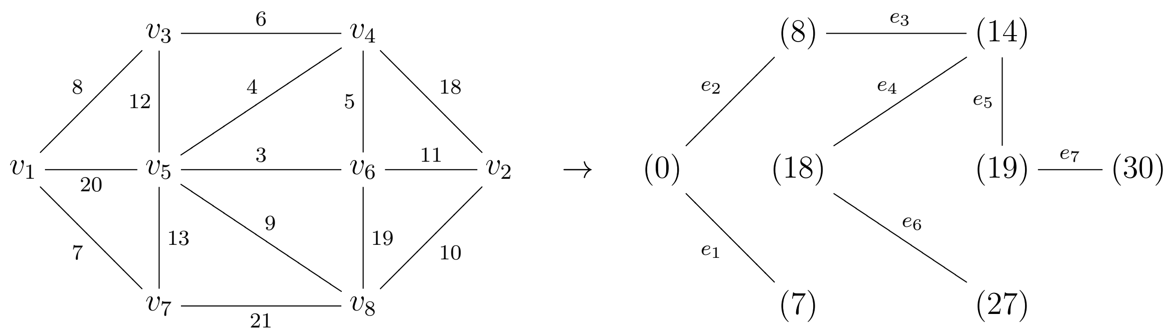
Algorithm 19 (Dijkstra)**.** For a chosen vertex v_k in a weighted graph G, we determine $d(v_k, x)$ for all $x \in V_G$, i.e., we get the entire k-th row of the distance matrix of G.

1. Denote by S the set of vertices s which have already been labeled by (W_s). Initially, we set $S = \{v_k\}$ and label it by $(W_{v_k} = 0)$.

2. For each vertex $x \in N(S)$, say adjacent to $y \in S$ whose edge xy has weight W, calculate the number $W_x = W_y + W$. Add to S the vertex x for which the corresponding W_x is least possible, and label x by (W_x). Hence, $W_x = d(a, x)$.

3. Repeat until all vertices have been labeled.

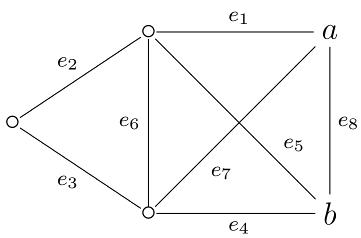
Example. We show Dijkstra's algorithm to evaluate $d(v_1, x)$ for each $x \in V_G$.



The distance matrix has Row (1) = $[0, 30, 8, 14, 18, 19, 7, 27]$, and the spanning tree shows the shortest path from v_1 to every other vertex. The spanning tree sequence $(7, 8, 6, 4, 5, 9, 11)$ resembles that of Prim's, but instead of choosing the least weight in each iteration, we consider the least accumulative weight relative to the starting vertex.

Definition. An *Euler walk* in a connected graph G is a walk through all the edges of G without repeating any edge. If an Euler walk is closed, we call it an *Euler circuit.*

Example. An Euler walk from a to b is the sequence $e_1, e_2, \ldots, e_8$:



Theorem 20. A connected graph has an Euler walk from a to b, with $a \neq b$, if and only if a and b are the only vertices of odd degree. The graph has an Euler circuit if and only if all vertices have an even degree.

Proof. Consider a vertex v with $\deg(v) = d$. At some point during the walk, we will run into v and out via another edge. If $d > 2$, this process will repeat, for as long as there are untrodden edges containing v. This shows the necessity that d be even, unless v is our starting vertex, or last, in which case d may be odd.

To prove sufficiency, assume first that every vertex in a graph G has an even degree. Consider a path P of maximum length from v to w. Since $\deg(w)$ is at least two, w is adjacent to another vertex already contained in P, else we could extend P to a longer path. Hence G contains a cycle C. Since every vertex in C has an even degree, so does the subgraph whose edges are in $G - C$. Repeating the argument, we see that the edges in G form the union of cycles none of which is disjoint from the rest.

We finish off by induction. One cycle is itself an Euler circuit. Assume that the union of n such cycles has an Euler circuit, call it E. With one more cycle C, which meets E at a vertex x, we have an Euler circuit for $E \cup C$ by starting at x, circuiting around E back to x, and cycling around C back to x.

Lastly, if $\deg(a)$ and $\deg(b)$ are the only odd degrees in G, we add one more edge, i.e., ab into G (perhaps making G a multigraph) so that every vertex now has an even degree. We have shown that an Euler circuit exists for this extended graph. Hence, without this extra edge, we could Euler walk from a and terminate at b. $\triangledown$

Definition. The *Chinese Postman Problem* asks for the minimal closed walk going through every edge in a weighted graph. If exists, an Euler circuit would certainly be the desired solution, otherwise such a walk would have to repeat one or more edges.

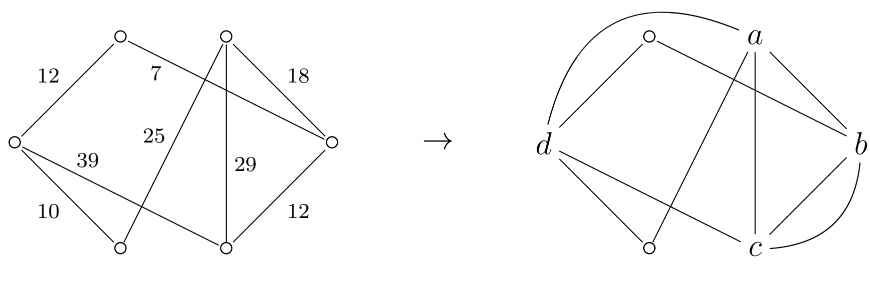
Algorithm 21 (Chinese Postman Problem)**.** We determine the minimal closed walk containing every edge in a given weighted graph.

1. Identify all vertices with odd degree. (By Euler's theorem, their number is even.)
2. Pair up these odd vertices two by two, say $\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_n, b_n\}$, in such a choice that minimizes the sum $\sum d(a_i, b_i)$.
3. Note that if we add "imaginary" edges $a_i b_i$, the resulting graph would have all vertices of even degree. Hence, solution to the Chinese postman problem is the Euler circuit on this imaginary graph, which is really a walk through all the edges of G plus the repetition of paths from a_i to b_i via the existing edges.

Example. The given graph has 4 vertices of odd degree: a, b, c, d. There are 3 possible pairings:

$$d(a,b) + d(c,d) = (18) + (12 + 7 + 12) = 49$$
$$d(a,c) + d(b,d) = (29) + (7 + 12) = 48$$
$$d(a,d) + d(b,c) = (25 + 10) + (12) = 47 \to \min$$

The Chinese Postman solution is a closed walk through all edges, plus the minimum repeated paths from a to d and from b to c. The total weight is $152 + 47 = 199$.



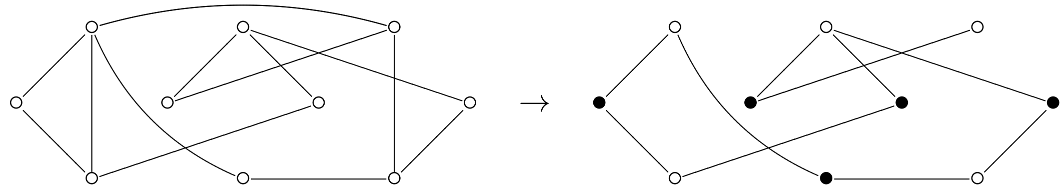
Definition. A *Hamilton cycle* in a graph G is a cycle $C_n \subseteq G$ such that $n = |V_G|$, i.e., a closed walk through all the vertices in G without repeating any vertex except, of course, the end vertex. If a Hamilton cycle exists, then G is called a *Hamilton graph*.

Example. K_4 and $K_{3,3}$ are Hamilton graphs. P_4 is not a Hamilton graph. Note that a Hamilton graph is necessarily connected.

Theorem 22. If G is a Hamilton graph, then G contains no leaves, no bridges, and no cut vertices. A *cut vertex* is one that disconnects the graph when removed.

Proof. A Hamilton cycle (or any cycle) is 2-regular, hence every vertex in G needs to have degree at least two. Now a bridge is the only path between two components, so a closed walk through both components must cross the bridge twice, hence not a cycle. Similarly, a closed walk through a cut vertex must repeat the vertex. ▽

Example. We can see that if a vertex has degree two, then any Hamilton cycle must contain both its edges. Using this fact, we prove that the given graph on the left is not Hamilton: There are five vertices of degree two, indicated on the right by black dots. So all the edges where these dots are on must be part of the Hamilton cycle, if exists.



However, the incomplete solution has a vertex of degree 3, so no Hamilton cycle (or any cycle) can have this subgraph.

Theorem 23. Let G be a connected graph with n vertices. If $\deg(v) \geq \frac{n}{2}$ for all $v \in V_G$, then G is a Hamilton graph.

Proof. Let $|V| = n$ and P be a path of maximum length contained in G, given by $v_1v_2, v_2v_3, \ldots, v_{k-1}v_k$. Having maximum length means that the neither v_1 nor v_k is adjacent to any other vertex outside P. Since $\deg(v_1) \geq \frac{n}{2}$, there are at least this many vertices in P adjacent to v_1, and similarly to v_k. By the pigeonhole principle, we can find v_j, with $2 \leq j \leq k$, such that both $v_1v_j, v_{j-1}v_k \in E$. This gives us a cycle given by the closed walk

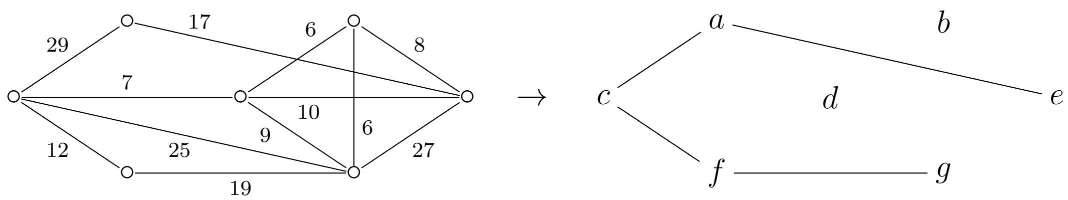
$$v_1v_2, \ldots, v_{j-2}v_{j-1}, v_{j-1}v_k, v_kv_{k-1}, \ldots, v_{j+1}v_j, v_jv_1$$

Any additional vertex connected to this cycle would contradict the maximality of the length of P. Hence $k = n$, i.e., we have a Hamilton cycle. ▽

Definition. The *Traveling Salesman Problem* asks for a Hamilton cycle of least total weight in a given weighted graph. One way to solve the problem is to try out all possible Hamilton cycles, which is reasonable only in small cases.

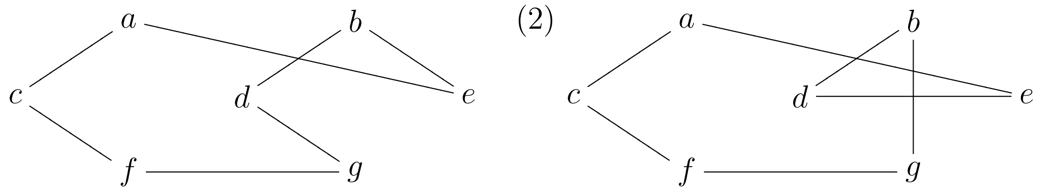
Algorithm 24 (Traveling Salesman Problem)**.** We find all Hamilton cycles of a given weighted graph in order to choose one with least total weight. We assume that the graph is relatively small or has mostly vertices of degree two.

Example. The given graph on the left has two vertices of degree two, labeled a and f as shown on the right with a sketch of the incomplete Hamilton cycle.



If we start from e and complete the cycle, there remain b and d. This gives 2! different Hamilton cycles, i.e., (1) $eacfg - db - e$ and (2) $eacfg - bd - e$.

(1) (2)



We check that the second Hamilton cycle has the lesser total weight of 99.

4 Coloring

Definition. The set notation $X \sqcup Y$ is the *disjoint union* of X and Y, i.e., the ordinary set union $X \cup Y$ but with the assumption that $X \cap Y = \emptyset$. A graph G is *bipartite* if $V_G = X \sqcup Y$ and two vertices can be adjacent only if they belong to opposite subsets. In other words, a bipartite graph is just a subgraph of $K_{m,n}$. We call X and Y the *bipartition subsets* of G.

Example. The cycle C_6 with edges $\{ab, bc, cd, de, ef, fa\}$ is bipartite with $X = \{a, c, e\}$ and $Y = \{b, d, f\}$. In this case, $C_6 \subset K_{3,3}$. Note that if G is disconnected, then G is bipartite if and only if each component is bipartite.

Theorem 25. If G is a closed walk of odd length, it contains a cycle of odd length.

Proof. A closed walk of length three is C_3, so the claim is true. We proceed by induction, assuming the theorem has been proved if the length is less than n. If G is the walk $v_1v_2, v_2v_3, \ldots, v_nv_1$ with no repeated vertex, then $G \simeq C_n$ and we are done. Suppose now $v_i = v_{i+j}$. Then G is the union of two closed walks: the one from v_i to v_{i+j}, and the walk from v_1 to v_i joined by that from v_{i+j} to v_1. One of the two must have an odd length because their sum is odd, and which is clearly less than n. By our hypothesis, that walk contains a cycle of odd length. $\triangledown$

Theorem 26. A graph G is bipartite if and only if there is no cycle of odd length.

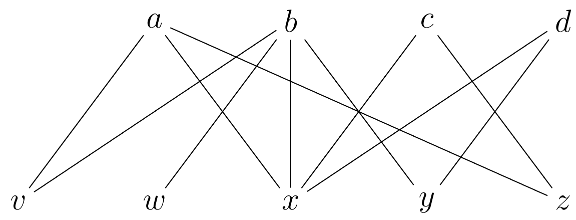
Proof. If G is bipartite and $v_1v_2, v_2v_3, \ldots, v_nv_1$ is a cycle, then by definition we deduce that $v_1, v_3, v_5, \ldots$ all belong to the same bipartition subset. In particular, $v_1v_n \in E_G$, so n must be even. Conversely, let G contain no odd cycle. We assume that G is connected, or else it suffices to consider a component of G. Fix a vertex v and set

$$X = \{w \in V_G \mid d(v, w) \text{ is even}\} \quad \text{and} \quad Y = \{w \in V_G \mid d(v, w) \text{ is odd}\}$$

It is clear that $V_G = X \sqcup Y$. Assume $a, b \in X$ and we will show $ab \notin E_G$. (The case $a, b \in Y$ is symmetrical). Now there is a path of even length from v to a and also to b. So if $ab \in E_G$, then the union of ab and these two paths makes a closed walk of odd length. Then by Theorem 25, G would have an odd cycle, a contradiction. $\triangledown$

Definition. Let G be a bipartite graph with $V_G = X \sqcup Y$, and assume from now on that $|X| \leq |Y|$ (otherwise, we swap X and Y). Every edge $ab \in E_G$ can be viewed as a relation $(a, b) \in X \times Y$. We say that G has a *complete matching* when we can find a set of these relations (i.e., edges) that form a one-to-one function from all of X into Y.

Example. A complete matching here can be $\{av, by, cz, dx\}$, or $\{az, bw, cx, dy\}$, etc.

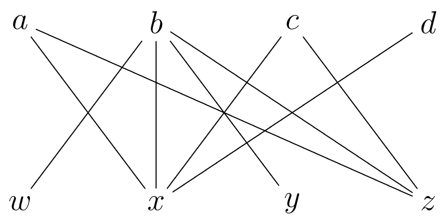


Theorem 27 (Hall's Theorem)**.** Let G be bipartite with $V_G = X \sqcup Y$. Then G has a complete matching if and only if $|S| \leq |N(S)|$ for every subset $S \subseteq X$.

Proof. The necessary condition $|S| \leq |N(S)|$ is required by the definition of one-to-one function. Now assume that this condition is satisfied and assume also, by induction, that we have a matching M contaning all of X except one vertex $v \in X$. Observe that if we have path $P = \{v_1v_2, v_2v_3, \ldots, v_{2n-1}v_{2n}\}$ such that $v_kv_{k+1} \in M$ if and only if k is even, and that both $v_1, v_{2n} \notin M$, then replacing the edges in $M \cap P$ by those in $P - M$ will produce a new matching M' which contains one more vertex from each X and Y. To complete the induction, we will produce such an "alternating" path from $v_1 = v$.

Since $N(v)$ is non-empty, we can find $w_1 \in Y$ which is adjacent to v. If $w_1 \notin M$ then vw_1 is such path, we are done. Else, there is an edge $v_1w_1 \in M$. Since $|N(\{v, v_1\})| \geq 2$, we have w_2, adjacent to either v or v_1. If $w_2 \notin M$, again we have an alternating path from v to w_2, so assume there is another edge $v_2w_2 \in M$. Continuing in this way, seeing that $|N(\{v, v_1, v_2, \ldots v_k\})| > k$ in each step, we will exhaust the vertices in $Y \cap M$, forcing a vertex $w \notin M$ to which there is an alternating path from v. ▽

Example. In the bipartite graph below, a complete matching is not possible because we have a counter-example with $S = \{a, c, d\}$, for which $|S| > |N(S)|$.



Theorem 28. Suppose that a regular graph G is bipartite with $V_G = X \sqcup Y$. Then $|X| = |Y|$ and G has a complete matching.

Proof. Assume that G is d-regular with $|X| = n$, so that $|E_G| = dn$. Now if $|Y| = m$, then we have $|E_G| = dm$, hence it is necessary that $m = n$. Furthermore, any set S of k vertices in X corresponds to dk edges which are connected to k vertices in Y. We then have $|N(S)| = |S|$ and a complete matching by Hall's theorem. ▽

Definition. The term *graph coloring* means assigning a color to every vertex in G in such a way that adjacent vertices have distinct colors. The *chromatic number* $\chi(G)$ stands for the least possible number of colors needed to perform this task.

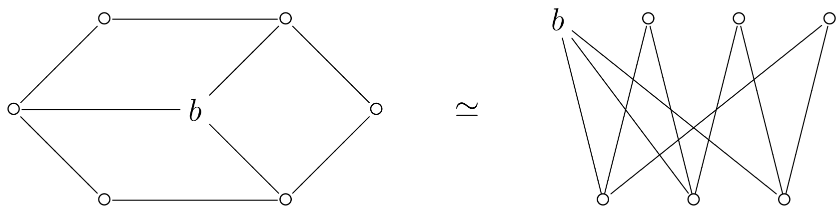
Note that $\chi(G) = 1$ if and only if G is trivial. If not trivial, then $\chi(G) = 2$ if and only if G is bipartite. It is clear that if $H \subseteq G$, then $\chi(H) \leq \chi(G)$. In particular, if G is disconnected, then $\chi(G)$ is simply the largest chromatic number among the components of G.

Example. We claim that $\chi(C_5) = 3$. To prove this, first we show that 3 colors are sufficient (demonstrate it), then we show that 2 colors do not suffice, e.g., since odd cycle is not bipartite.

Algorithm 29 (Bi-Coloring)**.** Given a graph G, we apply two colors to determine bipartite or not bipartite and if so, we produce the bipartition subsets.

1. Any vertex can be chosen to start with by assigning the color black.
2. Let S denote the set of vertices which have already been colored. Now color every uncolored vertex in $N(S)$ white or black as determined by their adjacent color. If this is not possible, then G is not bipartite.
3. Repeat until all have been colored, in which case the vertices are bipartitioned according to their colors, black or white.

Example. Below we start at vertex b (black) and successfully color all vertices black or white. Omitting details, the bipartition reconstructed on the right shows that the graph is bipartite.



Theorem 30. Let $\chi(G) = k$. Then the graph G has a vertex v such that $\deg(v) \geq k-1$, and at least k such vertices exist.

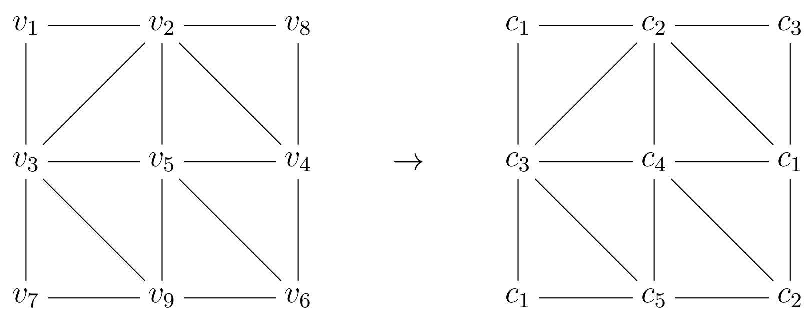
Proof. Let H be a minimal subgraph of G with $\chi(H) = k$, i.e., such that any subgraph of H will have chromatic number less than k. It is clear that H has at least k vertices. To complete the proof, we claim that in H we have $\deg(v) \geq k-1$ for all $v \in V_H$. This is true because by the choice of H, the subgraph of H minus a vertex v can be colored with $k-1$ colors. So if $\deg(v) \leq k-2$, then we can assign v one of these $k-1$ colors to complete coloring H, and we would have $\chi(H) \leq k-1$, a contradiction. $\triangledown$

Theorem 31. For any graph G, we have $\chi(G) \leq \Delta(G) + 1$.

Proof. By the theorem, there is $v \in V_G$ such that $\Delta(G) \geq \deg(v) \geq \chi(G) - 1$. $\triangledown$

Algorithm 32 (Sequential Coloring)**.** We color a graph G with pre-ordered vertices $V_G = \{v_1, v_2, \ldots, v_n\}$ as follows. Let $c_1, c_2, \ldots$ denote colors, and for $i = 1, 2, \ldots, n$, we assign to v_i the color c_m with the least possible value of m.

Example. The graph G below has 9 ordered vertices. The Sequential Coloring algorithm yields the color sequence $(1, 2, 3, 1, 4, 2, 1, 3, 5)$. These 5 colors may not be minimum, so $\chi(G) \leq 5$. The algorithm does not determine the chromatic number, and a different vertex ordering will probably result in a different number of colors.



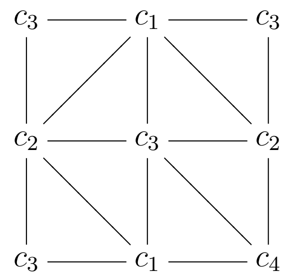
Algorithm 33 (Welsh-Powell Coloring)**.** We color the vertices $V_G = \{v_1, v_2, \dots, v_n\}$ by prioritizing vertices of larger degrees.

1. Re-order the vertices $w_1, w_2, \dots, w_n$ from the largest degree to the smallest.
2. For $i = 1, 2, \dots, n$, assign to w_i the color c_1 whenever possible.
3. For $i = 1, 2, \dots, n$, assign to w_i the color c_2, if yet uncolored, whenever possible, and repeat in like manner using $c_3, c_4, \dots$ until all have been colored.

Example. Using Welsh-Powell algorithm, we re-color the same graph from the previous example, showing the vertex re-ordering and its color assignment below:

v_i	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
$\deg(v_i)$	2	5	5	4	5	3	2	2	4
w_i	v_2	v_3	v_5	v_4	v_9	v_6	v_1	v_7	v_8
color(w_i)	c_1	c_2	c_3	c_2	c_1	c_4	c_3	c_3	c_3

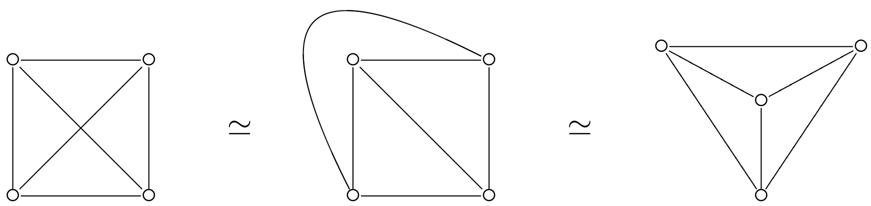
$\rightarrow$



The resulting color sequence is $(3, 1, 2, 2, 3, 4, 3, 3, 1)$, only 4 colors this time. Currently no known coloring algorithm can effectively determine $\chi(G)$.

Definition. A graph is *planar* if it can be drawn in the plane such that no edges are crossing each other. This particular drawing of a planar graph is called a *plane graph*.

Example. We show two ways of drawing the graph K_4 without crossing edges:

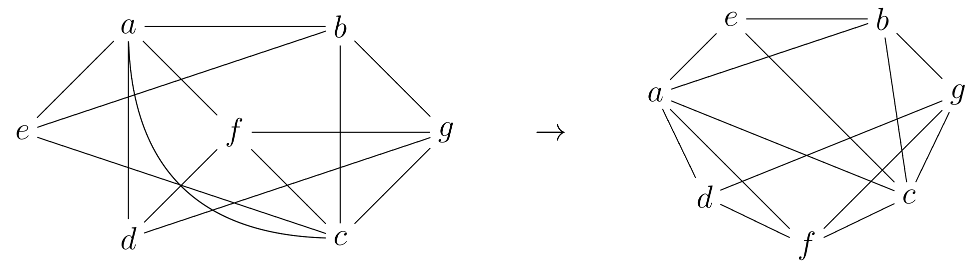


Similarly, we can show that $K_{2,3}$ is planar and later, that K_5 and $K_{3,3}$ are not. Note that if G is disconnected, then G is planar if and only if each component is planar.

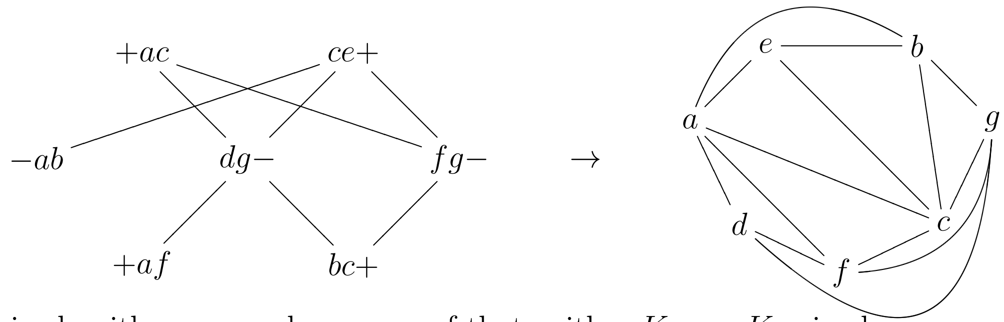
Algorithm 34 (Planarity Test for Hamilton Graphs)**.** Given a Hamilton graph G, we determine planar or not planar.

1. Draw a Hamilton cycle H in a shape somewhat circular.
2. Transform H into G by adding the remaining edges $e_1, e_2, \dots, e_m$, all of which are drawn interior to the cycle H.
3. Let K be the graph with vertices $V_K = \{e_1, e_2, \dots, e_m\}$ such that e_i, e_j are adjacent in K if and only if they are crossing each other in H. Hence, the number of edges in K is the number of intersections inside H.
4. Determine whether K is bipartite or not bipartite. The graph G is planar if and only if K is bipartite.

Example. The given graph G is on the left. First, we find the Hamilton cycle $aebgcfda$ and draw the remaining edges inside this cycle.



The resulting graph K has 7 vertices (from the edges inside H) and 8 edges (from the intersections inside H). K is proved bipartite by 2-coloring, so G is planar. We draw the plane graph after moving the (–) edges outside of H while keeping the (+) inside.



With this algorithm we can have a proof that neither K_5 nor $K_{3,3}$ is planar.

Definition. A plane graph partitions the plane into subsets which are called *regions*. In other words, regions refer to the bounded areas interior to the plane graph, plus one unbounded exterior.

Example. The plane graph of K_4, pictured earlier, has four regions.

Theorem 35 (Euler's Formula)**.** Suppose that a connected plane graph has V vertices, E edges, and R regions. Then

$$V + R = E + 2$$

Proof. If the graph is a tree, then $E = V - 1$ and $R = 1$ as there is no bounded region. Hence, the identity $V + R = E + 2$ holds in a tree. A graph that is not a tree can be constructed by adding edges to its spanning tree. Note that each additional edge will add one to the bounded regions, hence Euler's formula is preserved. ▽

Theorem 36 (Euler's Planarity Test)**.** Suppose that G is connected and that $V \geq 3$.

1. If $E > 3V - 6$, then G is not planar.

2. If $E > 2V - 4$ and G has no triangles, then G is not planar.

Proof. Given a fixed number of edges, the number of regions is maximized when every region is the interior of a triangle. (With higher polygons, a diagonal edge can be added while maintaining planarity.) Since every edge borders two regions, this maximum R is given by the relation $R_{\max} = \frac{2E}{3}$. Then, by Euler's formula,

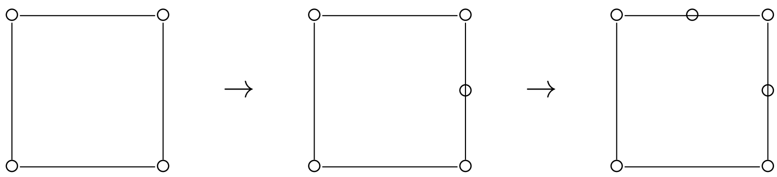
$$E = V + R - 2 \leq V + \frac{2E}{3} - 2$$

which simplifies to $E \leq 3V - 6$. This inequality holds for planar graphs, hence Euler's planarity test is justified. In cases where G has no triangles, we will have $R_{\max} = \frac{2E}{4}$, and the claim will follow similarly. ▽

Example. The graph K_5 has 10 edges, where $10 > 3(5) - 6$, hence not planar. As for $K_{3,3}$, we have 9 edges, 6 vertices, and no triangles (bipartite has no odd cycle), hence not planar since $9 > 2(6) - 4$.

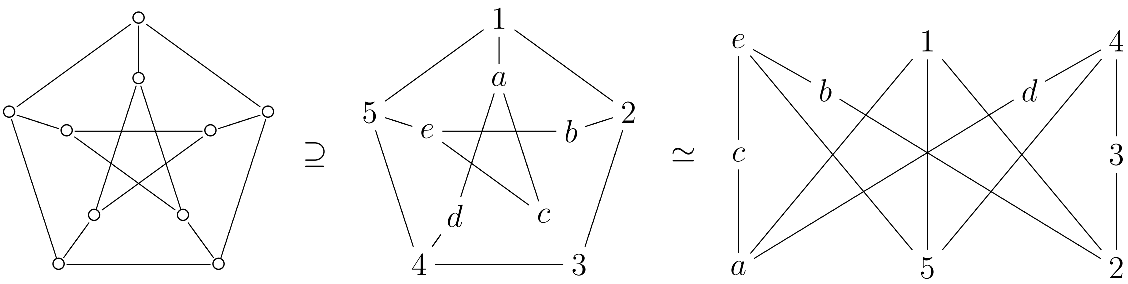
Definition. If we replace any edge $ab \in E_G$ by the path $\{av, vb\}$, where v is a new vertex, then the resulting graph is said to be *homeomorphic* to G. Now *homeomorphism* is an equivalence relation among graphs in which two of them are homeomorphic if one can be obtained from the other by iterating a finite number of replacements in this manner.

Example. We sketch below how to obtain C_6 by applying the procedure twice to C_4. In this way, it is not hard to see that any two cycles are homeomorphic.



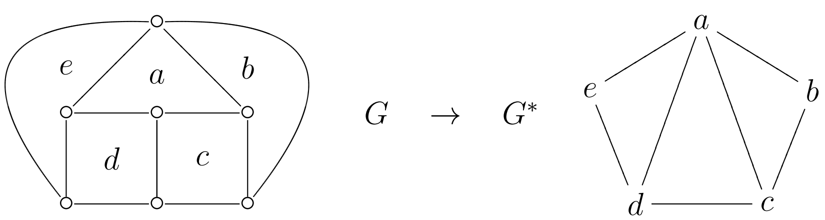
Theorem 37 (Kuratowski's Theorem)**.** A graph is planar if and only if it contains no subgraph homeomorphic to K_5 or $K_{3,3}$.

Example. The *Peterson graph* is the graph shown on the left. We can prove that the Peterson graph is not planar by producing a subgraph which, after a little modification in the way it is drawn, is shown to be homeomorphic to $K_{3,3}$.



Definition. The *dual graph* G^* of a planar graph G is the graph whose vertices are the interior regions from the plane graph of G, and they are adjacent if and only if the regions are bounded by a common edge.

Example. We show below a plane graph G with five interior regions, next to its dual graph G^* with these five as vertices.



Theorem 38. The dual graph of any plane graph is planar. Conversely, every planar graph is the dual graph of some plane graph.

Proof. For each region r of the plane graph G, put the vertex $r \in V_{G^*}$ right inside this region. So for two neighboring regions, the edge rs can be drawn as a curve that crosses the edge bordering the two regions and, by definition, only this edge. Hence the edges in G^* can be drawn without crossing one another, and this procedure can also be done in reverse. ▽

Definition. A plane graph somewhat looks like a world map in which the interior regions represent countries. We say two countries are neighbors when they share a common edge boundary. (Having a common vertex does not make a neighbor.) In fact, map coloring was an early motivation for planar graphs.

We define the *chromatic number* of a map to be the least number of colors enough to color the countries such that neighboring countries have distinct colors. In other words, the chromatic number of a map G is given by $\chi(G^*)$.

Theorem 39. If G is planar, then $\chi(G) \leq 6$.

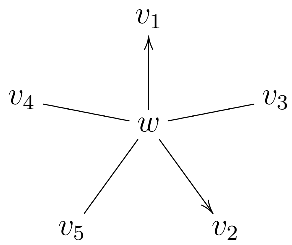
Proof. Suppose that G is planar with n vertices. We claim that there exists a vertex $w \in V_G$ such that $\deg(w) \leq 5$; If this were false, we would have $\deg G \geq 6n$ and then, by Euler's theorem, $|E_G| \geq 3n$, contradicting Euler's planarity test.

We proceed by induction on n. Since $n \leq 6$ is trivially true, we assume the theorem holds for $n-1$ vertices. In particular, the subgraph of G obtained by removing the vertex w and its associated edges is planar, hence can be colored with 6 colors. We may extend this coloring for all of G by assigning a color to w. Since w has at most 5 neighbors, at least one color from those 6 can be chosen for w, giving us $\chi(G) \leq 6$. $\triangledown$

Theorem 40. If G is planar, then $\chi(G) \leq 5$.

Proof. If G has 5 vertices or less, there is nothing to prove. Again by induction, we assume that G has n vertices and that the claim is true for any planar graph with less vertices than n. As in the preceding proof, G has a vertex w with $\deg(w) \leq 5$. By the *color-degree* of a vertex we mean the number of distinct colors of vertices which are adjacent to it. Now because the subgraph G minus w can be colored with at most 5 colors, if the color-degree of w is 4 or less, then we easily assign to w one of the 5 colors to complete the inductive step.

The last case is when $\deg(w) = 5$ with all five adjacent vertices, v_1, v_2, v_3, v_4, v_5, having distinct colors c_i. Without loss of generality, we have labeled these vertices such that v_3 is interior of the region bounded by the rays wv_1 and wv_2, while v_4 lies exterior of it.



Consider the subgraph $G_{1,2}$ of G consisting of all vertices which have been colored either c_1 or c_2, with their associated edges. Note that $v_1, v_2 \in G_{1,2}$. Suppose first that v_1 and v_2 belong to different components in $G_{1,2}$. In the component containing v_1, we swap c_1 and c_2. Doing so does not violate the rules of vertex coloring, but it does decrease the color-degree of w to 4 as v_1 and v_2 now have the same color, i.e., c_2.

But if v_1 and v_2 belong to the same component, then we have a path from v_1 to v_2 which becomes a cycle when combined with v_2w and wv_1. Since G is planar, this cycle must enclose either v_3 or v_4, but not both. Again, since G is planar, v_3 and v_4 then belong to different components in the subgraph $G_{3,4}$, defined in a similar way. This time, we decrease the color-degree of w by swapping c_3 and c_4 in the component containing v_3, and the proof is complete. $\triangledown$

Theorem 41 (The Four-Color Theorem)**.** If G is planar, then $\chi(G) \leq 4$. Consequently, every map is colorable using only four colors or less.

Exercises

Chapter (1)

1. Find $|E_G|$.
   (a) K_{27} (b) $K_{10,32}$ (c) P_{56} (d) C_{77}

2. Find $\Delta(G)$.
   (a) K_{50} (b) $K_{24,45}$ (c) P_{99} (d) C_{87}

3. Find $\deg G$.
   (a) K_{18} (b) $K_{22,15}$ (c) P_{68} (d) C_{91}

4. A complete graph has 351 edges. Find the number of vertices.

5. G is a complete bipartite graph with $\deg G = 144$ and $\Delta(G) = 9$. Find $|V_G|$.

6. If G is d-regular, determine d.
   (a) K_{99} (b) $K_{99,99}$ (c) P_{99} (d) C_{99}

7. Draw an example of a 3-regular graph with degree 18.

8. A graph is 5-regular with 35 edges. Find the number of vertices.

9. A complete bipartite graph is regular with 12 vertices. Find the number of edges.

10. Determine the degree sequence.
    (a) K_5 (b) $K_{2,3}$ (c) P_6 (d) C_6

11. A graph has degree sequence $(6, 5, 4, 3, 3, 2, 1, 1, 1)$. Find the number of edges.

12. Determine graphical or not graphical. If graphical, draw the graph.
    (a) $(3, 2, 2, 1, 1, 1)$ (b) $(4, 3, 3, 2, 1, 0)$ (c) $(5, 3, 2, 2, 1, 1)$ (d) $(5, 4, 4, 3, 3, 3, 3, 2, 2, 1)$

13. Draw two different examples of a graph with the given degree sequence.
    (a) $(4, 4, 3, 2, 2, 1)$ (b) $(2, 2, 2, 2, 2, 1, 1)$ (c) $(5, 3, 2, 2, 1, 1, 1, 1)$ (d) $(3, 3, 3, 3, 3, 3, 3, 3)$

14. Determine whether or not G contains a cycle.
    (a) $K_{3,2}$ (b) $K_{1,6}$ (c) P_9 (d) C_8

15. Find all values of n such that $C_n \subseteq K_{3,4}$.

16. Draw an example where G and $\overline{G}$ are connected and $\Delta(\overline{G}) \neq \Delta(G)$.

17. A graph with 9 vertices is disconnected. What is the maximum number of edges?

18. Determine how many bridges in G.
    (a) $K_{11,2}$ (b) $K_{1,14}$ (c) P_{25} (d) C_{61}

19. Draw an example of a connected graph such that every edge is a bridge, and the complement is also connected but without any bridge.

20. Draw the complement of G.
    (a) K_5 (b) $K_{3,4}$ (c) C_5 (d) P_5

21. Given the degree sequence of G, determine the degree sequence of $\overline{G}$.
(a) $(4,4,3,2,2,1)$ (b) $(5,3,2,2,1,1,1,1)$ (c) $(8,6,6,6,5,4,3,3,1)$ (d) P_5

22. Given G, find $\deg(\overline{G})$.
(a) P_{10} (b) C_7 (c) $K_{4,4}$ (d) $K_{1,9}$

23. A graph is self-complementary with 17 vertices. Find the number of edges.

24. Draw three different examples of a self-complementary graph.

25. Determine the adjacency matrix.
(a) K_5 (b) $K_{3,2}$ (c) P_5 (d) C_5

26. Given the adjacency matrix, find $\deg(G)$ and $\deg(\overline{G})$.

$$(a)\ \begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}\quad (b)\ \begin{bmatrix}0&1&1&1&0\\1&0&1&1&1\\1&1&0&1&1\\1&1&1&0&1\\0&1&1&1&0\end{bmatrix}\quad (c)\ \begin{bmatrix}0&1&0&1&1\\1&0&0&0&1\\0&0&0&1&1\\1&0&1&0&1\\1&1&1&1&0\end{bmatrix}\quad (d)\ \begin{bmatrix}0&1&0&1&0&1\\1&0&1&0&1&0\\0&1&0&1&0&1\\1&0&1&0&1&0\\0&1&0&1&0&1\\1&0&1&0&1&0\end{bmatrix}$$

27. Let A denote the adjacency matrix for C_4 and B the adjacency matrix for $K_{2,2}$. Find a permutation matrix P such that $A = PBP^T$.

28. Determine the incidence matrix.
(a) K_4 (b) $K_{3,2}$ (c) P_5 (d) C_5

29. Determine how many rows and columns in the incidence matrix of K_{20}.

30. Given the incidence matrix, determine the adjacency matrix.

$$(a)\ \begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}\quad (b)\ \begin{bmatrix}1&0\\0&1\\1&0\\0&1\end{bmatrix}\quad (c)\ \begin{bmatrix}1&0&0&0\\1&1&1&0\\0&1&0&1\\0&0&1&1\end{bmatrix}\quad (d)\ \begin{bmatrix}1&0&0&0&0&1&1&0\\1&0&1&0&1&0&0&0\\0&1&0&1&1&0&0&0\\0&1&1&0&0&0&1&1\\0&0&0&1&0&1&0&1\end{bmatrix}$$

Chapter (2)

1. Determine G is a tree or not a tree.
(a) $K_{32,2}$ (b) $K_{1,66}$ (c) P_{95} (d) C_{42}

2. Given G, determine $\overline{G}$ is cyclic or acyclic.
(a) $K_{2,2}$ (b) $K_{1,4}$ (c) P_4 (d) C_6

3. Draw three different examples of a tree with 6 vertices.

4. Find the degree of a tree with 20 vertices.

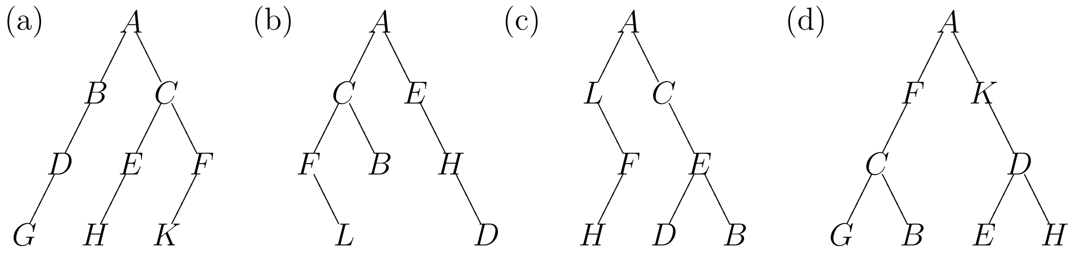
5. Find the number of edges of a tree that is self-complementary.

6. Determine whether or not G contains a leaf.
(a) $K_{3,2}$ (b) $K_{1,6}$ (c) P_9 (d) C_8

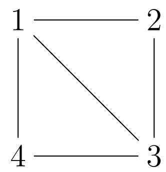
7. Given the degree sequence, determine G is a tree or not a tree.
   (a) $(5, 3, 2, 2, 1, 1)$ (b) $(5, 2, 2, 1, 1, 1)$ (c) $(5, 2, 1, 1, 1, 1, 1)$ (d) $(5, 3, 2, 1, 1, 1, 1)$

8. A tree has degree 20 and two vertices of degree 5. Find the number of leaves.

9. Given the labeled binary tree, determine the traversal sequence using (i) pre-order (ii) post-order (iii) in-order algorithm.

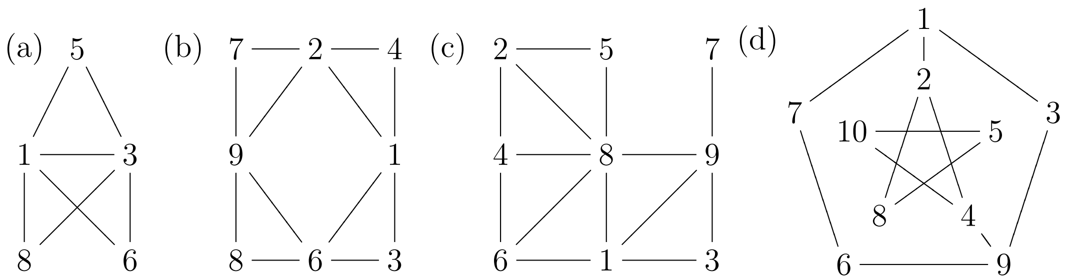


10. Draw all the spanning trees of the given graph.

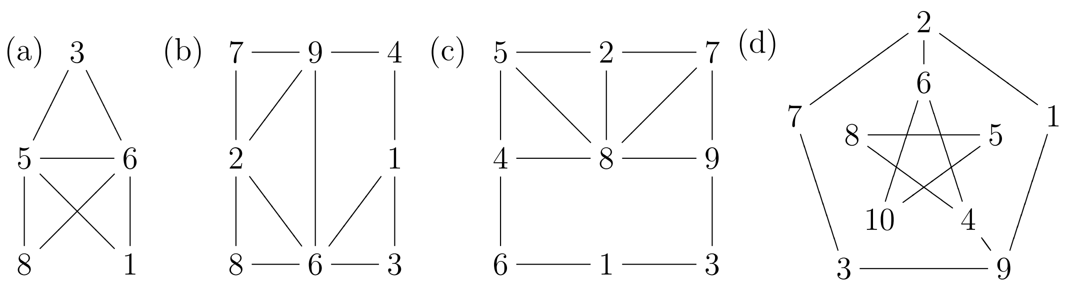


11. Determine the number of spanning trees using a cofactor of $D - A$.
    (a) K_4 (b) $K_{2,3}$ (c) P_4 (d) C_5

12. Draw the DFS spanning tree starting at vertex (1) and write the DFS sequence.



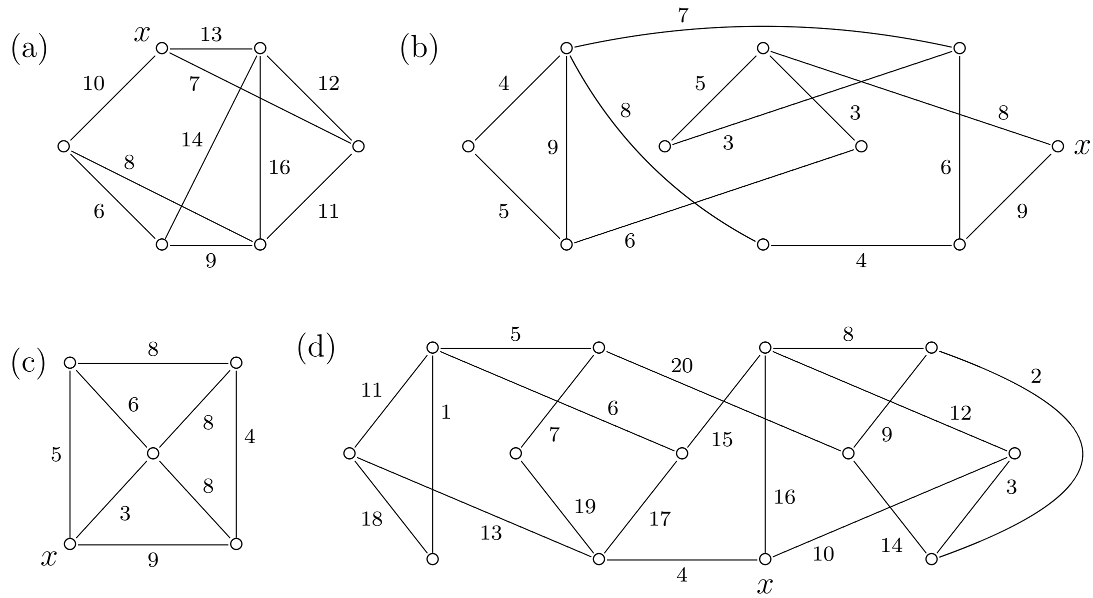
13. Draw the BFS spanning tree starting at vertex (1) and write the BFS sequence.



14. Given the weight matrix, find the MST sequence and its total weight using (i) Kruskal's algorithm (ii) Prim's algorithm starting at vertex (1).

(a) $\begin{bmatrix} 0 & 12 & 6 & 10 \\ 12 & 0 & 4 & 3 \\ 6 & 4 & 0 & 5 \\ 10 & 3 & 5 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 9 & 0 & 14 & 12 \\ 9 & 0 & 8 & 0 & 7 \\ 0 & 8 & 0 & 6 & 5 \\ 14 & 0 & 6 & 0 & 4 \\ 12 & 7 & 5 & 4 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 0 & 3 & 0 & 5 \\ 1 & 0 & 3 & 0 & 5 & 0 \\ 0 & 3 & 0 & 5 & 0 & 7 \\ 3 & 0 & 5 & 0 & 7 & 0 \\ 0 & 5 & 0 & 7 & 0 & 9 \\ 5 & 0 & 7 & 0 & 9 & 0 \end{bmatrix}$

15. Draw Prim's minimal spanning tree starting at (x) and write the MST sequence.



Chapter (3)

1. Determine the number of triangles in the graph using adjacency matrix.
(a) K_4 (b) $K_{2,2}$ (c) $K_{2,3}$ (d) K_5

2. Determine the number of triangles in K_{50} without using adjacency matrix.

3. Determine whether or not G contains a triangle.
(a) $K_{1,5}$ (b) $K_{2,4}$ (c) $K_{3,3}$ (d) $K_{5,2}$

4. Determine the distance matrix.
(a) K_5 (b) $K_{3,2}$ (c) P_5 (d) C_5

5. Given the adjacency matrix, determine the distance matrix and the diameter.

$$(a)\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad (b)\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \quad (c)\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \quad (d)\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

6. Find $d(G)$.
(a) K_{99} (b) $K_{99,99}$ (c) P_{99} (d) C_{99}

7. Find a formula for $d(G)$.
(a) K_n (b) $K_{m,n}$ (c) P_n (d) C_n

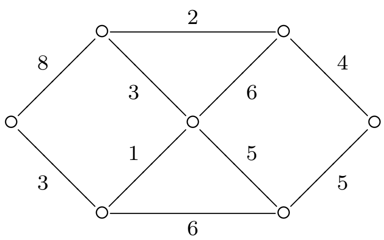
8. Find $d(\overline{C_6})$.

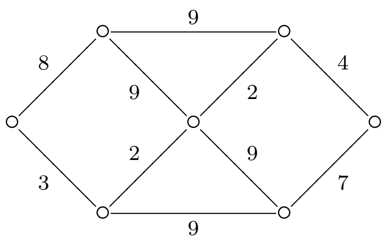
9. Draw an example of a graph with 7 vertices, diameter 4, and no leaf.

10. Given the weight matrix, determine the distance matrix and the diameter.

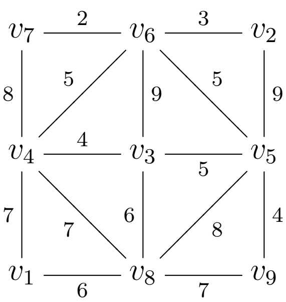
(a) $\begin{bmatrix} 0 & 12 & 6 & 10 \\ 12 & 0 & 4 & 3 \\ 6 & 4 & 0 & 5 \\ 10 & 3 & 5 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 12 & 0 & 14 & 3 \\ 12 & 0 & 8 & 0 & 4 \\ 0 & 8 & 0 & 9 & 5 \\ 14 & 0 & 9 & 0 & 7 \\ 3 & 4 & 5 & 7 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 3 & 1 & 3 & 1 \\ 1 & 0 & 1 & 3 & 1 & 3 \\ 3 & 1 & 0 & 1 & 3 & 1 \\ 1 & 3 & 1 & 0 & 1 & 3 \\ 3 & 1 & 3 & 1 & 0 & 1 \\ 1 & 3 & 1 & 3 & 1 & 0 \end{bmatrix}$

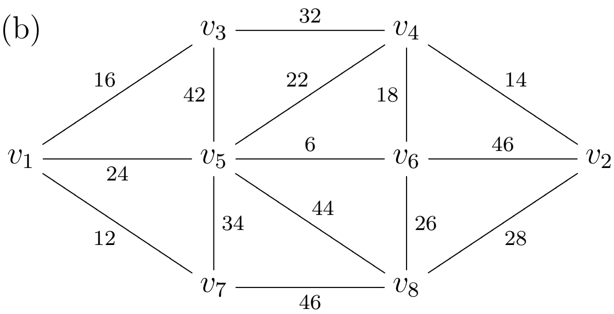
11. Given the weighted graph, determine the diameter.

(a) 

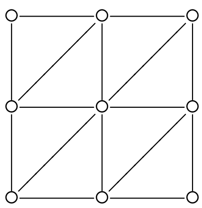
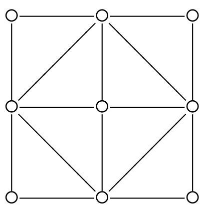
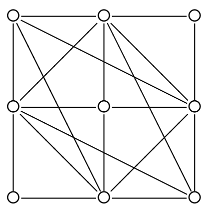
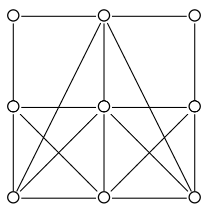
(b) 

12. Determine Row (1) and Row (5) of the distance matrix using Dijkstra's algorithm, and write the spanning tree sequence for each case.

(a) 

(b) 

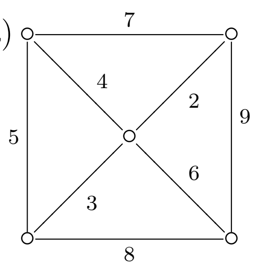
13. Find an Euler walk or Euler circuit in the graph, if exists.

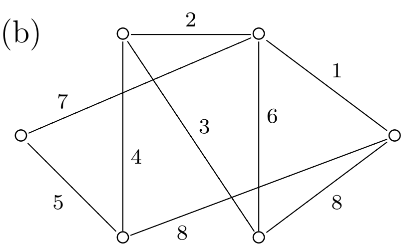
(a)  (b)  (c)  (d) 

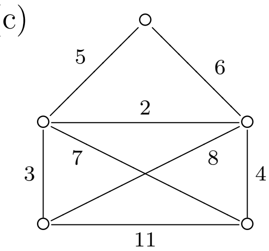
14. Determine whether or not G has an Euler walk/circuit.
(a) $K_{4,4}$ (b) $K_{5,4}$ (c) $K_{9,9}$ (d) $K_{9,2}$

15. Given G, determine whether or not $\overline{G}$ has an Euler walk/circuit.
(a) P_{11} (b) P_{14} (c) C_{12} (d) C_{13}

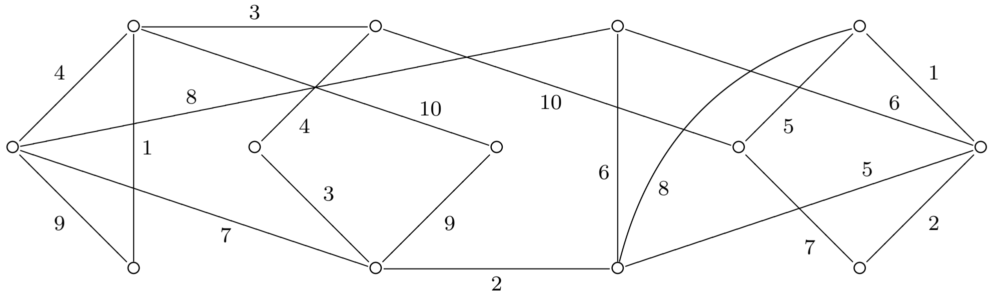
16. Solve the Chinese Postman problem.

(a) 

(b) 

(c) 

(d)

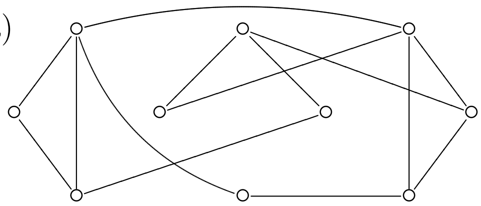


17. Determine G is Hamilton or not Hamilton.
    (a) K_{99} (b) $K_{99,1}$ (c) $K_{99,2}$ (d) $K_{99,99}$

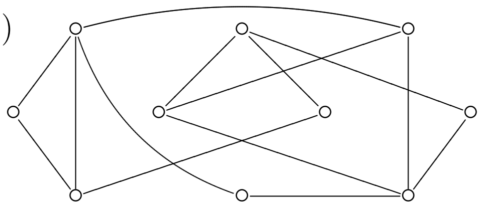
18. Determine if $\overline{C_6}$ is Hamilton or not Hamilton.

19. Determine how many Hamilton cycles exist and draw them.

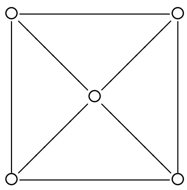
(a)



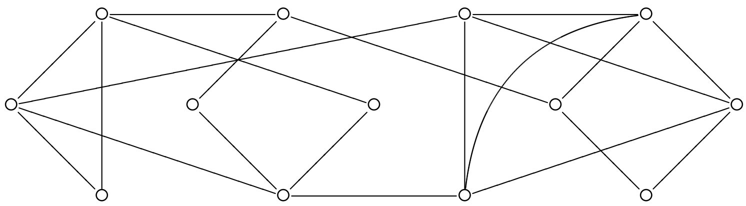
(b)



(c)

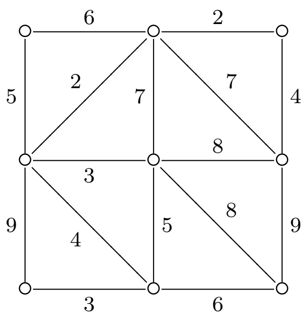


(d)

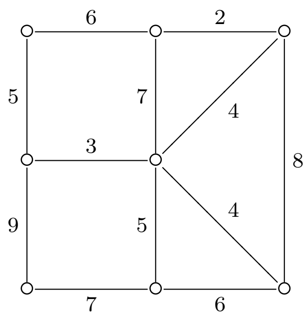


20. Solve the Traveling Salesman problem by drawing all possible Hamilton cycles.

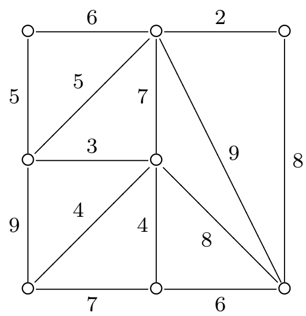
(a)



(b)



(c)



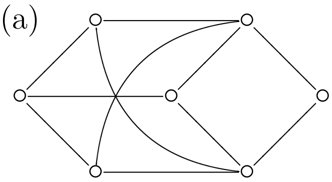
Chapter (4)

1. Determine G is bipartite or not bipartite.
   (a) C_{39} (b) C_{48} (c) P_{27} (d) P_{56}

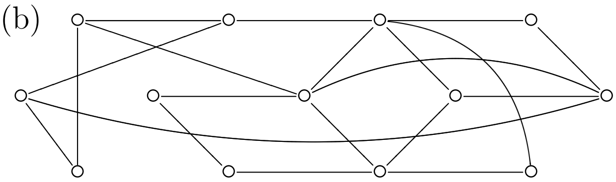
2. Determine $\overline{C_6}$ is bipartite or not bipartite.

3. If G is bipartite, draw it as a subset of $K_{m,n}$. If not bipartite, find an odd cycle.

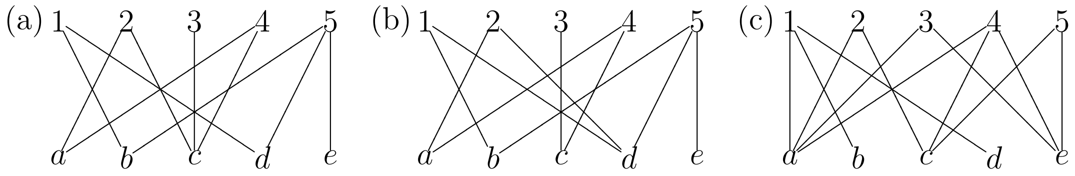
(a)



(b)



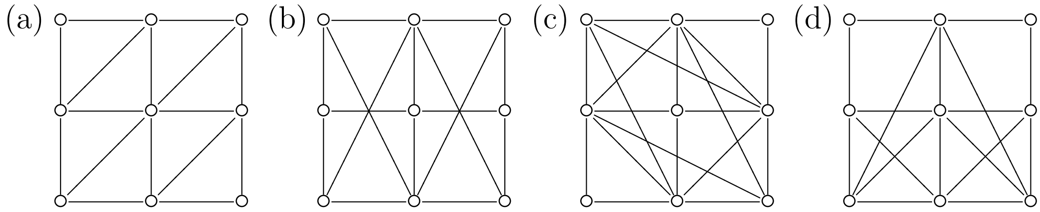
4. Find a complete matching or prove not exist.



5. Determine $\chi(G)$.
   (a) P_{99} (b) C_{99} (c) P_{100} (d) C_{100}

6. Find a formula for $\chi(G)$.
   (a) K_n (b) $K_{m,n}$ (c) P_n (d) C_n

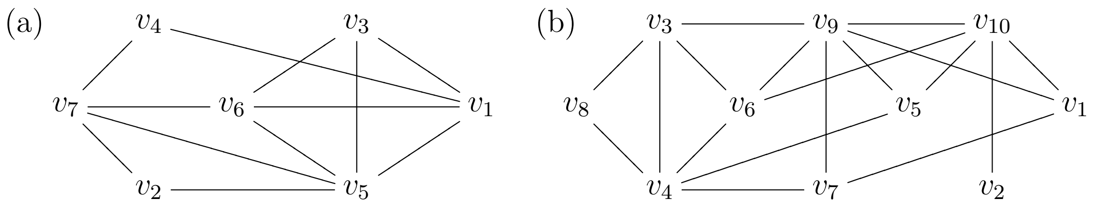
7. Find the chromatic number for each graph.



8. Find the chromatic number of $\overline{C_6}$.

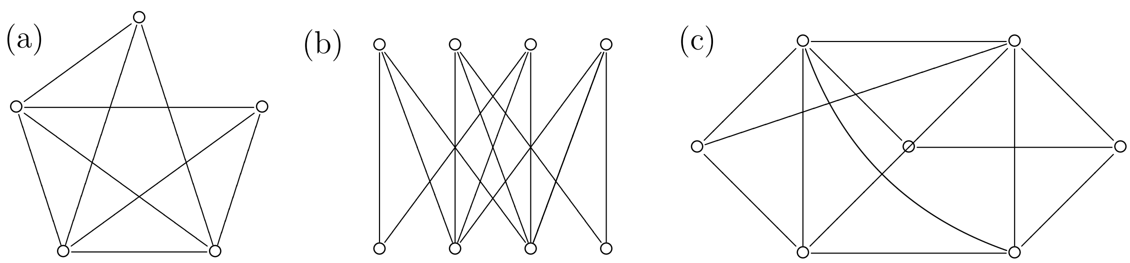
9. Which one applies? (A) $\chi(G) = \Delta(G) + 1$ (B) $\chi(G) = \Delta(G)$ (C) $\chi(G) < \Delta(G)$
   (a) K_n (b) $K_{m,n}$ (c) P_n (d) C_n

10. Color the graph using Sequential Coloring algorithm and write the color sequence.



11. Repeat Problem (10) using Welsh-Powell algorithm.

12. Prove planar or not planar using Hamilton cycle. If planar, draw the plane graph.



13. Determine $\overline{C_6}$ is planar or not planar.

14. A planar graph has 10 vertices and degree 32. Find the number of regions.

15. Draw an example of a plane graph with 8 vertices and 8 regions.

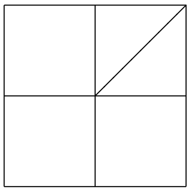
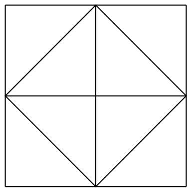
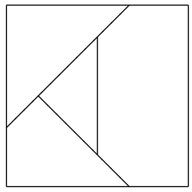
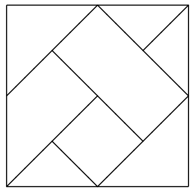
16. A bipartite planar graph has degree 42. What is the maximum number of regions?

17. Use Euler's planarity test to prove G is not planar, if it applies.
    (a) $\overline{P_8}$ (b) $\overline{C_6}$ (c) $\overline{P_7}$ (d) $\overline{C_7}$

18. Draw the dual graph of $K_{2,9}$.

19. Given the dual graph, find an example of the map.
    (a) K_4 (b) P_5 (c) $K_{2,3}$ (d) C_5

20. Find the chromatic number of the map by coloring its dual graph.

(a)  (b)  (c)  (d) 

Extra Problems

1. Prove that in every graph there exist two vertices of equal degree.
2. Describe the complement of $K_{m,n}$ in general.
3. Prove that $\overline{C_n}$ is connected for all $n \geq 5$.
4. Find all examples of a connected graph G with 5 vertices such that $\overline{G}$ is also connected, with $\Delta(G) \neq \Delta(\overline{G})$.
5. Prove that if G is self-complementary, then $|E_G| = \frac{1}{4}n(n-1)$, where $n = |V_G|$.
6. Let $V_G = \{v_1, v_2, \ldots, v_n\}$ with adjacency matrix A. Prove that $[A^2]_{ii} = \deg(v_i)$.
7. Prove that adding an edge to a tree will produce a cycle.
8. Prove that if $|E_G| \geq |V_G|$, then the graph G is cyclic.
9. Prove that $K_{m,n}$ is a tree if and only if it has a leaf.
10. Prove that $K_{1,n}$ is the only tree with disconnected complement.
11. Prove that P_4 is the only self-complementary tree.
12. Let G be a tree with $|V_G| \geq 3$ and n leaves. Prove that $K_n \subseteq \overline{G}$.
13. Given a degree sequence $(d_1, d_2, \ldots, d_n)$ with $d_n \neq 0$ and $d_1 + \cdots + d_n = 2n - 2$, prove there exists a tree with this degree sequence.
14. Determine the number of spanning trees for P_n and C_n in general.
15. Prove that there are n^{n-2} different labeled trees with vertices $v_1, v_2, \ldots, v_n$, where $n \geq 2$, by showing that n^{n-2} is the number of spanning trees of a labeled K_n.
16. Prove that if G is self-complementary, then $d(G) = 2$ or 3.
17. Can two non-isomorphic graphs have the same distance matrix?

18. Prove that $d(\overline{C_n}) = 2$ for all $n \geq 5$.
19. Prove that P_n is the only tree that has an Euler walk.
20. Determine all the values of n for which $\overline{C_n}$ has an Euler walk.
21. Let p be a prime number. Prove that C_p is the only connected regular graph with p edges.
22. Prove that $\overline{C_n}$ is a Hamilton graph for all $n \geq 5$.
23. Consider the weighted graph given by the weight matrix in Problem (3.10b). Find the minimum total weight of a closed walk through all the vertices in the graph. (Hint: the minimum closed walk here is not given by a Hamilton cycle.)
24. Let G be a bipartite graph with n vertices. Prove that $|E_G| \leq \frac{1}{4}n^2$ and in particular, $|E_G| = \frac{1}{4}n^2$ if and only if $G \simeq K_{\frac{n}{2},\frac{n}{2}}$.
25. Prove that $\overline{C_n}$ is not bipartite for all $n \geq 5$.
26. Let G be a bipartite graph with $V_G = X \sqcup Y$. Prove that if G is a Hamilton graph, then $|X| = |Y|$.
27. Let G be a connected graph. Prove that G is a 2-regular bipartite graph if and only if G is a cycle of even length.
28. Let G be connected with three or more leaves. Prove that $\overline{G}$ is not bipartite.
29. Prove that $K_{m,n}$ has a complete matching if and only if it is a Hamilton graph.
30. Prove that $\chi(G) = 2$ if G is a non-trivial tree.
31. Prove that $\chi(\overline{C_n}) \geq k$ for all $n \geq 2k$.
32. Find an example of a graph with chromatic number 4 that does not contain K_4.
33. Consider the graph as pictured in Problem (3.11a). Find a particular vertex ordering for the Sequential Coloring algorithm that will result in 4 colors.
34. Prove that $\overline{C_n}$ is not planar for all $n \geq 7$.
35. Prove that a d-regular graph is not planar for all $d \geq 6$.
36. Prove that if G is a planar graph with at least 11 vertices, then $\overline{G}$ is not planar.
37. Show that in general $K_{2,n}$ is planar by drawing the plane graph, then determine its dual graph.
38. Prove that every graph G is homeomorphic to a bipartite graph, if we replace every edge in G by a path of length two.
39. The Peterson graph is given in the Example following Theorem 37. Determine (a) the degree sequence (b) the degree (c) regular or irregular (d) the diameter (e) Euler or not Euler (f) bipartite or not bipartite (g) the chromatic number (h) all its C_n subgraphs.
40. The Peterson graph is neither planar nor Hamilton. Determine whether or not its complement is planar or is Hamilton.