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INFINITE COUNTABLE SETS

Amin Witno

Abstract

We introduce the cardinal numbers in association with the cardinality of an arbitrary set and the ordering relation which leads to the important theorem of Cantor and the continuum hypothesis.

These notes are written to supplement Homework Set #10 in the Set Theory course (Math 251) at Philadelphia University, Jordan. Outline notes are more like a revision. No student is expected to fully benefit from these notes unless they have regularly attended the lectures.[1]

Cardinality

For any set A, not assumed finite, we denote by $|A|$ what we call the *cardinality* of the set A. With this, we define $C = \{|A| \mid \emptyset \subseteq A\}$ and call the elements of C *cardinal numbers*. For instance, $|\{0\}|$, $|\{0, 1, 2\}|$, $|\{3, 5, 7\}|$, and $|\mathbb{Z}|$ are some examples of cardinal numbers.

Note that the proposition $\emptyset \subseteq A$ is a tautology, hence it applies to any set A there is. For this reason, the set C is properly refered to as the set of *all* cardinal numbers.

We shall construct three binary relations on the set C, and study their properties, in the following order.

Definition. Let A and B be two arbitrary sets, again not necessarily finite. We shall write $|A| = |B|$ if there exists a bijection $f : A \to B$. Moreover, we define the relation $R^= = \{(|A|, |B|) \mid |A| = |B|\}$.

Before we proceed, observe that this definition agrees with the ordinary meaning of cardinality for finite sets:

Proposition 1. Let A and B be two finite sets, and let $|A|$ and $|B|$ denote the number of elements in A and B, respectively. Prove that $|A| = |B|$ if and only if there exists a bijection $f : A \to B$.

Proof. In class. ▽

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awitno@gmail.com

Exercise 2. Suppose that $|A| = |B|$, where A and B are two finite sets, and consider any function $f : A \to B$. Show that f is injective if and only if f is surjective.

Theorem 3. The relation $R^{=}$ is an equivalence relation on C.

Proof. We have $|A| = |A|$ because the identity function $i(a) = a$ for all $a \in A$ is definitely bijective. This proves that $R^{=}$ is reflexive. Next, suppose that $|A| = |B|$ and let $f : A \to B$ be a bijection. Then $f^{-1} : B \to A$ is again a bijection, so $|B| = |A|$ and $R^{=}$ is symmetric. Finally, for transitive, let $|A| = |B|$ and $|B| = |C|$. It follows that there exist bijections $f : A \to B$ and $g : B \to C$, whose composition $g \circ f : A \to C$ is again a bijection, proving that $|A| = |C|$. $\bigtriangledown$

Having established this theorem, we shall now identify every cardinal number with its equivalence class with respect to the equivalence relation $R^{=}$. In other words, from now on when we write $|A|$, we really mean the class $[|A|] = \{|B| \mid |B| = |A|\}$. Hence, we consider the cardinal numbers $|A|$ and $|B|$ identical if $|A| = |B|$, i.e., if there is a bijection $f : A \to B$. This also explains why the equal sign "=" has been chosen to denote the relation.

For example, we will not distinguish the cardinal number $|\{0, 1, 2\}|$ from the cardinal number $|\{3, 5, 7\}|$. As a consequence, the set C of cardinal numbers is now "reduced", containing only cardinal numbers which are distinct one from another—*distinct* here refers to $|A|$ and $|B|$ for which $(|A|, |B|) \notin R^{=}$, and for which we may write $|A| \neq |B|$.

Definition. Let us denote the cardinality of the set of natural numbers by $\aleph_0$—to be read *aleph naught*. Thus, $\aleph_0 = |\mathbb{N}|$.

For example, observe that the function $f(n) = n + 1$ is a bijection from $\mathbb{N}$ onto $\{2, 3, 4, \ldots\}$ and consequently, $|\mathbb{N} - \{1\}| = |\mathbb{N}| = \aleph_0$. This is totally unacceptable if the sets involved were finite, for a finite set cannot possibly have a cardinality equal to that of its proper subset.

Exercise 4. Consider any finite set $S \subseteq \mathbb{N}$. Prove that $|\mathbb{N} - S| = \aleph_0$.

Furthermore, if we consider the functions $f(n) = 2n$ and $f(n) = 2n-1$, respectively, then we may conclude that the cardinality of the set of even natural numbers, and similarly for the odd ones, are both $\aleph_0$, despite the fact that either set contains infinitely less than what the set $\mathbb{N}$ has. Even more surprisingly, the set of all integers also has cardinality equal to that of positive integers:

Theorem 5. We have $|\mathbb{Z}| = \aleph_0$.

Proof. Define the function $f : \mathbb{Z} \to \mathbb{N}$ by $f(n) = 2n$ if $n > 0$ and by $f(n) = -2n + 1$ if $n \leq 0$. In other words, the positive integers are mapped onto the even numbers, while zero and the negative integers are mapped onto the odd numbers. It is not hard to verify that f is a bijection. $\bigtriangledown$

Exercise 6. Suppose that $|A| = |B| = \aleph_0$. Show that $|A \cup B| = \aleph_0$.

We will come back to this quantity $\aleph_0$ after we introduce and become familiar with the second binary relation on C, next.

Cantor

Definition. We shall write $|A| \leq |B|$ to mean that there exists an injection $f : A \to B$. With this notation, we define the relation $R^{\leq}$ on C by $R^{\leq} = \{(|A|, |B|) \mid |A| \leq |B|\}$.

Because a bijection is also an injection, we have $R^{=} \subseteq R^{\leq}$. In other words, if we have $|A| = |B|$, then we also have $|A| \leq |B|$. Moreover, as before, we shall show that this definition again coincides with the meaning of cardinality for finite sets:

Proposition 7. Let A and B be two finite sets, and let $|A|$ and $|B|$ denote the number of elements in A and B, respectively. Prove that $|A| \leq |B|$ if and only if there exists an injection $f : A \to B$.

Proof. In class. $\bigtriangledown$

Exercise 8. With finite sets, prove that $|A| \leq |B|$ if and only if there exists a surjection $f : B \to A$.

Theorem 9. The relation $R^{\leq}$ on C is both reflexive and transitive.

Proof. For any set A, the identity function on A is an injection, hence $|A| \leq |A|$. It is also clear that the composition of two injections is again an injection, i.e., if $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$ by way of composition. $\bigtriangledown$

Exercise 10. If $A \subseteq B$, prove that $|A| \leq |B|$.

We shall see next that $R^{\leq}$ is also anti-symmetric, which therefore makes $R^{\leq}$ a partial order relation. It is generally accepted that Cantor was the one who proposed this fact, without providing a proof. The first complete proof was presented by Bernstein and another, independently, by Schroeder.

Theorem 11. Let A and B be two arbitrary sets such that $|A| \leq |B|$ and $|B| \leq |A|$. Then $|A| = |B|$.

To prove the theorem, we let $f : A \to B$ and $g : B \to A$ be a pair of injections. We shall construct a bijection $F : A \to B$ following a series of steps in this order:

1. Let $A_0 = A$ and $B_0 = B$. We define the sets A_n and B_n for $n \geq 1$ inductively as follows. Set $A_1 = g(B_0)$ and $B_1 = f(A_0)$. Then for $n \geq 2$ we let $A_n = g(B_{n-1})$ and $B_n = f(A_{n-1})$. We claim that $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ and similarly also $B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots$

2. Let $S_n = A_n - A_{n+1}$ and $T_n = B_n - B_{n+1}$ for all $n \geq 0$. From the previous step, we see that the sets S_n are disjoint one from another, and similarly for T_n as well. We claim that for all $n \geq 0$, we have $f(S_n) = T_{n+1}$, hence $|S_n| = |T_{n+1}|$; and similarly $g(T_n) = S_{n+1}$, hence $|T_n| = |S_{n+1}|$.

3. We next define
$$S = A - \bigcup_{n \geq 0} S_n \quad \text{and} \quad T = B - \bigcup_{n \geq 0} T_n$$
Thus we partition the set A into the disjoint subsets $S, S_0, S_1, S_2, \ldots$, and similarly B into $T, T_0, T_1, T_2, \ldots$ We claim that $f(S) = T$, hence $|S| = |T|$.

The function $F : A \to B$ is finally constructed by setting

$$F(a) = \begin{cases} f(a) & \text{if } a \in S_n \text{ and } n \text{ is even} \\ g^{-1}(a) & \text{if } a \in S_n \text{ and } n \text{ is odd} \\ f(a) & \text{if } a \in S \end{cases}$$

That F is a bijection follows from the fact that F is put together from the pieces of bijections between these partitioning subsets of A and B.

Proof of (1). It is clear that $A_1 \subseteq A_0$ and $B_1 \subseteq B_0$. Furthermore, $A_2 = g(B_1) \subseteq g(B_0) = A_1$ and $B_2 = f(A_1) \subseteq f(A_0) = B_1$. We proceed by induction. Since $A_n = g(B_{n-1})$ and $A_{n+1} = g(B_n)$, the hypothesis $B_n \subseteq B_{n-1}$ implies that $g(B_n) \subseteq g(B_{n-1})$, i.e., $A_{n+1} \subseteq A_n$. Similarly, if $A_n \subseteq A_{n-1}$, then $f(A_n) \subseteq f(A_{n-1})$ and $B_{n+1} \subseteq B_n$. ▽

Proof of (2). The fact that $A_{n+1} \subseteq A_n$ implies that $f(A_n - A_{n+1}) = f(A_n) - f(A_{n+1})$. Hence $f(S_n) = B_{n+1} - B_{n+2} = T_{n+1}$. Similarly, we also have $g(T_n) = g(B_n) - g(B_{n+1}) = A_{n+1} - A_{n+2} = S_{n+1}$. ▽

Proof of (3). First let $x \in S$ and we will show that $f(x) \in T$, hence $f(S) \subseteq T$. By definition, if $x \in S$ then $x \notin S_n$ for any $n \geq 0$. Since $f(S_n) = T_{n+1}$ and f is injective, we conclude that $f(x) \notin T_n$ for any $n \geq 1$. It is clear that also $f(x) \notin T_0$ because $T_0 = B - f(A)$ and $x \in A$. It follows that $f(x) \in T$.

Now let $y \in T$ and we will show that $f(x) = y$ for some $x \in S$, in order to prove that $f(S) = T$. By definition, if $y \in T$ then $y \notin T_n$ for any $n \geq 0$. Since $T_n = B_n - B_{n+1}$, we conclude that $y \in B_n$ for all $n \geq 1$, i.e., that $y \in f(A_n)$ for all $n \geq 0$. It follows that, since f is injective, there is a *unique* $x \in A$ such that $f(x) = y$ and such that $x \in A_n$ for all $n \geq 0$, hence $x \notin S_n$ for any $n \geq 0$ and so $x \in S$. ▽

In fact, not only that $R^{\leq}$ is a partial ordering, but $R^{\leq}$ is furthermore a total order relation on C. What this means is that given any two sets A and B, either $|A| \leq |B|$ or $|B| \leq |A|$ must hold. (If both hold, of course, then $|A| = |B|$.) However, this advanced result is beyond our range of study. Instead later, in Theorem 15 we will establish a weaker fact, i.e., a special case in which $B = \mathbb{N}$. For now, we can use Theorem 11 to prove another amazing fact: that the set of rational numbers and the set of natural numbers have the same cardinality.

Theorem 12. We have $|\mathbb{Q}| = \aleph_0$.

Proof. Since it is obvious that $|\mathbb{N}| \leq |\mathbb{Q}|$, using Theorem 11 we will be done if we can demonstrate that $|\mathbb{Q}| \leq |\mathbb{N}|$. Let $A = \mathbb{Z} \times \mathbb{N}$. Every rational number in its reduced form can be expressed as m/n with $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. This gives a natural injection which implies that $|\mathbb{Q}| \leq |A|$. Furthermore, by Theorem 5 we can find a bijection from $\mathbb{Z}$ onto $\mathbb{N}$ which by extension becomes a bijection that gives $|A| = |\mathbb{N} \times \mathbb{N}|$. Hence, by transitivity, we are left to showing that $|\mathbb{N} \times \mathbb{N}| \leq |\mathbb{N}|$. We do this via the function $f(m, n) = 2^m \times 3^n$, which is an injection for if $2^m \times 3^n = 2^{m'} \times 3^{n'}$, then $m = m'$ and $n = n'$ by the uniqueness of prime factorization. ▽

Exercise 13. Suppose that $|A| = |B| = \aleph_0$. Prove that $|A \times B| = \aleph_0$.

It was Cantor again who introduced Theorem 12 and proved it. Of course, Cantor did not use Theorem 11 to establish Theorem 12 because he had not proved the former. Instead, Cantor showed that the rational numbers can be ordered $a_1, a_2, a_3, \ldots$ to make the one-to-one correspondence with the natural numbers. To illustrate how this is done, it suffices to consider only the positive numbers—partially listed in the following matrix.

1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$	$\frac{11}{2}$	$\frac{13}{2}$	$\frac{15}{2}$	$\frac{17}{2}$	$\frac{19}{2}$	$\frac{21}{2}$	$\cdots$
		$\frac{1}{3}$	$\frac{2}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{7}{3}$	$\frac{8}{3}$	$\frac{10}{3}$	$\frac{11}{3}$	$\frac{13}{3}$	$\frac{14}{3}$	$\cdots$
			$\frac{1}{4}$	$\frac{3}{4}$	$\frac{5}{4}$	$\frac{7}{4}$	$\frac{9}{4}$	$\frac{11}{4}$	$\frac{13}{4}$	$\frac{15}{4}$	$\frac{17}{4}$	$\cdots$
				$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{6}{5}$	$\frac{7}{5}$	$\frac{8}{5}$	$\frac{9}{5}$	$\cdots$
					$\frac{1}{6}$	$\frac{5}{6}$	$\frac{7}{6}$	$\frac{11}{6}$	$\frac{13}{6}$	$\frac{17}{6}$	$\frac{19}{6}$	$\cdots$
						$\ddots$						

Observe that the i-th row contains all reduced fractions of the form a/i, in increasing order as we go from left to right. The counting is done columnwise, left to right:

$$1, 2, \frac{1}{2}, 3, \frac{3}{2}, \frac{1}{3}, 4, \frac{5}{2}, \frac{2}{3}, \frac{1}{4}, 5, \frac{7}{2}, \frac{4}{3}, \frac{3}{4}, \frac{1}{5}, 6, \cdots$$

e.g., the twentieth rational number in this ordering would be $2/5$. In this way, it is clear that every positive rational number a/i has a definite place in the matrix, say the k-th place in the sequence, giving the bijection $f(a/i) = k$ onto $\mathbb{N}$.

Countability

Definition. Let us call a set A *countable* if $|A| \leq \aleph_0$. In other words, A is countable when there exists an injection $f : A \to \mathbb{N}$.

The following are several examples to illustrate this definition of countability.

1. The set $\mathbb{N}$ itself is countable by the reflexivity $|\mathbb{N}| \leq |\mathbb{N}|$.
2. Every subset of a countable set is again countable. For if $S \subseteq A$, then $|S| \leq |A|$ according to Exercise 10. If in addition A is countable, i.e., if $|A| \leq \aleph_0$, then $|S| \leq \aleph_0$ by transitivity.
3. Every finite set is countable, for if $A = \{a_1, a_2, \ldots, a_n\}$, then $f(a_k) = k$ is an injection from A into $\mathbb{N}$.
4. The set $\mathbb{Z}$ is countable, according to Theorem 5.
5. The set $\mathbb{Q}$ is countable, according to Theorem 12.

Exercise 14. Revisit Exercises 6 and 13. Explain why the union and the cross product of two countable sets are also countable.

Having seen that $|A| \leq |\mathbb{N}|$ for every finite set A, we shall also demonstrate the converse: If A is an infinite set, then $|\mathbb{N}| \leq |A|$. For if A is infinite, we can find an infinite sequence of distinct elements of A, denoted by $a_1, a_2, a_3, \ldots$, then consider $f(k) = a_k$ for an injection from $\mathbb{N}$ into A.

In particular, when A is countable, i.e., $|A| \leq \aleph_0$, as well as infinite, i.e., $\aleph_0 \leq |A|$ by the preceding argument, then we will have $|A| = \aleph_0$ by Theorem 11. If A is finite, and though $|A| \leq \aleph_0$, we will never have $|A| = \aleph_0$ since $\mathbb{N}$ is infinite. We wish to be able to express this last fact by writing $|A| < \aleph_0$, after the following definition.

Definition. Let the relation $R^{<}$ on C be defined by $R^{<} = R^{\leq} - R^{=}$. Equivalently, $(|A|, |B|) \in R^{<}$ if and only if $|A| \leq |B|$ but $|A| \neq |B|$, in which case we shall write $|A| < |B|$. Alternately, we may write $|B| > |A|$ instead of $|A| < |B|$.

With this definition, we are able to state as a nice theorem the results already discussed in the preceding paragraphs, as follows.

Theorem 15. For any set A, exactly one of the following three mutually exclusive properties must hold.

1. A is finite (hence countable) and $|A| < \aleph_0$.
2. A is infinite, countable, and $|A| = \aleph_0$.
3. A is infinite, not countable, and $|A| > \aleph_0$.

Thus, there are exactly two kinds of countable sets: finite sets and those with cardinality $\aleph_0$. Moreover, there are exactly two kinds of infinite sets: those with cardinality $\aleph_0$ and those which are not countable—this *uncountable* latter kind is formally introduced as follows.

Definition. We call a set A *uncountable* when $|\mathbb{N}| < |A|$, i.e., when $|A| > \aleph_0$.

Theorem 15 assures that being uncountable is truly the negation of being countable, in the mathematical as well as in the English sense. Up to now we have not seen an example of an uncountable set. To find a first example, we will use the next theorem.

Theorem 16. For any set A, we have $|A| < |P(A)|$.

Proof. The function $f(a) = \{a\}$ is clearly an injection from A into $P(A)$, hence we have $|A| \leq |P(A)|$. Now let $g : A \to P(A)$ be any injection. To prove that $|A| \neq |P(A)|$, we must show that $g(A) \neq P(A)$, i.e., we will produce an element $S \in P(A) - g(A)$. Simply let $S = \{x \in A \mid x \notin g(x)\}$. Then $S \subseteq A$, hence $S \in P(A)$. Now for all $a \in A$, we have $a \in S$ if and only if $a \notin g(a)$. Hence it is impossible to have $g(a) = S$, and so $S \notin g(A)$. $\triangledown$

Hence, in particular, we may conclude that the power set of the natural numbers is uncountable, since $|P(\mathbb{N})| > |\mathbb{N}| = \aleph_0$. Another example of an uncountable set is given by the real numbers:

Theorem 17. The set $\mathbb{R}$ is uncountable.

Proof. By transitivity, it suffices to show that $|P(\mathbb{N})| \leq |\mathbb{R}|$. We do this by constructing the injection $f : P(\mathbb{N}) \to \mathbb{R}$ defined by $f(S) = \sum_{i \in S} 10^{-i}$ for every $S \subseteq \mathbb{N}$. To illustrate, we have $f(\{1, 2, 3\}) = 0.111$ and $f(\mathbb{N}) = 1/9$. It is clear that f is indeed an injection and we need only remark that the series given by $f(S)$ is convergent, e.g., by comparison test with the geometric series $\sum_{i \geq 1} 10^{-i}$. $\triangledown$

Continuum

Definition. Following Cantor, having shown that $|\mathbb{R}| > \aleph_0$, let us set $|\mathbb{R}| = c$ and call this cardinal number c *the cardinality of the continuum.*

In the proof of Theorem 17, and combined with Theorem 16, we see the ordering $\aleph_0 < |P(\mathbb{N})| \leq c$. We will now show that actually $P(\mathbb{N})$ also has the cardinality of the continuum.

Theorem 18. We have $|P(\mathbb{N})| = c$.

Proof. With Theorem 11, we need only show that $|\mathbb{R}| \leq |P(\mathbb{N})|$. Consider the real interval $I = [0,1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$. First we shall demonstrate that $|I| \leq |P(\mathbb{N})|$, as follows. Every $x \in I$ can be uniquely expressed in the form $x = \sum_{i \geq 1} a_i \times 2^{-i}$, where $a_i \in \{0,1\}$. (This is none other than the binary representation of the number $x \in [0,1]$. In particular, $x = 0$ if and only if $a_i = 0$ for all $i \geq 1$, and $x = 1 = 0.\overline{1}$ if and only if $a_i = 1$ for all $i \geq 1$.) It is then clear that we have an injection $f : I \to P(\mathbb{N})$ given by $f(x) = \{i \in \mathbb{N} \mid a_i = 1\}$, e.g., $f(0) = \emptyset$, $f(\frac{1}{2}) = \{1\}$, and $f(1) = \mathbb{N}$.

Next, we shall show that $|\mathbb{R}| \leq |I|$, so the theorem will follow by transitivity. We know from Calculus that $f(x) = \tan x$ is a bijection from the interval $(-\pi/2, \pi/2)$ onto $\mathbb{R}$. With scaling and transposition, we get the bijection $F(x) = \tan(\pi x - \pi/2)$ from the interval $(0,1)$ to $\mathbb{R}$. In all, we have the ordering $|\mathbb{R}| = |(0,1)| \leq |[0,1]| \leq |P(\mathbb{N})|$ and the proof is complete. $\bigtriangledown$

Exercise 19. Find a suitable bijection to show that the positive real interval $(0, \infty) = \{x \in \mathbb{R} \mid x > 0\}$ is uncountable with $|(0,\infty)| = c$.

The so-called *Cantor's continuum hypothesis* assumes that there is no cardinal number strictly between $\aleph_0$ and c, i.e., that there is no set A for which $|\mathbb{N}| < |A| < |\mathbb{R}|$. As an advanced result, it has been demonstrated by Cohen that Cantor's continuum hypothesis is independent from the rest of the common axioms of set theory.

Conclusion

We have established that $R^{\leq}$ is a partial order relation on the set of cardinal numbers and is in fact a well ordering, although we do not seek to prove the latter claim. Nevertheless, we observe that cardinal numbers do not always obey the same laws of ordering as natural numbers do. Loosely speaking, we have the following laws concerning the first infinite number $\aleph_0$. Here, note that n is understood a natural number and that we employ $2^{\aleph_0}$ to stand for $|P(\mathbb{N})|$.

1. $\aleph_0 \pm n = \aleph_0$
2. $\aleph_0 + \aleph_0 = \aleph_0$
3. $\aleph_0 \times n = \aleph_0$
4. $\aleph_0 \times \aleph_0 = \aleph_0$
5. $\aleph_0^n = \aleph_0$
6. $\aleph_0 < 2^{\aleph_0} < 2^{2^{\aleph_0}} < \cdots$